Delta Domain Riccati Equation——A Unified Approach*

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Abstract: By using the delta operator, the continuous and discrete Riccati matrix equation can be unified. Trace bounds and matrix bound for the solution of the unified algebraic Riccati equation are presented in this paper. The proposed results are less conservative than existing bounds and reduce to some new bounds separately for the continuous and discrete Riccati equations in the limiting cases.

Key words: Riccati equation; discrete time system; delta operator; digital control

Delta域Riccati方程研究:连续与离散的统一方法

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摘要: 基于Delta算子研究连续Riccati方程和离散Riccati方程的统一形式,得到Delta域Riccati方程解的定界估计.本文结果与现有结果相比,具有较小的保守性,在极限情形下可分别得到连续和离散Riccati方程的相关结论.

关键词: Riccati方程; 离散系统; Delta算子; 数字控制

1 Introduction

Riccati equation and Lyapunov equation are important in the fields of control theory, state estimation and signal processing. Some control and estimation problems such as optimal control, H_∞ control and Kalman filtering are often required to solve these equations. However, its exact solutions may be difficult to obtain when the dimension of the matrices involved increases. Therefore, the estimation problem for the solution of the Riccati and Lyapunov equations has attracted considerable attention in the past two decades. Bounds for the solution of Riccati equation provide approximations of the solution or initial estimates in the numerical algorithms for the exact solution. Many approaches for dealing with this topic was surveyed in [1].

Delta operator theory is widely used in high speed signal processing and digital control[2]. By the introduction of the delta operator, the continuous and discrete algebraic Riccati equation can be simultaneously studied in an unified framework. Ref. [3] derives eigenvalue bounds of the unified algebraic Riccati equation. In this paper, trace bounds for the solution of the unified algebraic Riccati equation are proposed.

2 Notation and preliminaries

$M = \{m_{ij}\} \in \mathbb{R}^{n\times n}$, M is a real matrix and M^{T} is the transpose of M. $M > 0$ ($M \geqslant 0$) denotes M is symmetric positive definite (positive semidefinite). $\mathrm{tr}(M)$ is the trace of M. I is the identity matrix. $\lambda_i(M)$ is an eigenvalue of M; Re $\lambda_i(M)$ is the real part of $\lambda_i(M)$ $|\lambda_i(M)|$ is the absolute of $\lambda_i(M)$; $\lambda_i(M)$, Re$\lambda_i(M)$ and $|\lambda_i(M)|$ are arranged in decreasing order, respectively, $i = 1,2,\cdots,n$, i.e. $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \cdots \geqslant \lambda_n(M)$; Re $\lambda_1(M) \geqslant \mathrm{Re}\lambda_2(M) \geqslant \cdots \geqslant \mathrm{Re}\lambda_n(M)$; $|\lambda_1(M)| \geqslant \cdots \geqslant |\lambda_n(M)|$.

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Consider the unified description of linear systems which covers continuous and discrete time cases simultaneously[4]:

$$\rho x(t) = Ax(t) + Bu(t). \tag{1}$$

where ρ is the generalized derivative which denotes d/dt (continuous case) or δ (discrete case). δ is called delta operator and defined by $\delta = (q-1)/\Delta$, Δ denotes the sampling period and q is the shift operator.

Now consider the delta operator formulated unified algebraic Riccati equation (UARE)

$$A^{\mathrm{T}}P + PA + \Delta A^{\mathrm{T}}PA - (\Delta A + I)^{\mathrm{T}}PB \cdot (I + \Delta B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(\Delta A + I) + Q = 0. \tag{2}$$

where $A, P, Q \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $P, Q > 0$ and $BB^{\mathrm{T}} \geqslant 0$. The matrix $(A - BB^{\mathrm{T}}P)$ is an asymptotically stable matrix, i.e. $\mathrm{Re}\lambda_i(A - BB^{\mathrm{T}}P) + \frac{1}{2}\Delta \mid \lambda_i(A - BB^{\mathrm{T}}P) \mid^2 < 0$[4].

Remark 1 From the unified algebraic Riccati equation (UARE), the continuous and discrete Riccati equations can be obtained in limiting cases, respectively. when $\Delta = 0$, follows the continuous algebraic Riccati equation (CARE); when $\Delta = 1$ with $A + I$ replaced by A, follows the discrete algebraic Riccati equation (DARE).

Lemma 1[5] If $X, Y \in \mathbb{R}^{n\times n}$, $X = X^{\mathrm{T}}$, $Y = Y^{\mathrm{T}}$, then for $1 \leqslant i, j \leqslant n$,

$$\lambda_{i+j-1}(X+Y) \leqslant \lambda_j(X)\lambda_i(Y),\quad i+j \leqslant n+1, \tag{3}$$

$$\lambda_{i+j-n}(X+Y) \geqslant \lambda_j(X)\lambda_i(Y),\quad i+j \geqslant n+1. \tag{4}$$

Lemma 2[5] If $X, Y \in \mathbb{R}^{n\times n}$, $X = X^{\mathrm{T}} \geqslant 0$, $Y = Y^{\mathrm{T}} \geqslant 0$, then for $1 \leqslant i, j \leqslant n$,

$$\lambda_{i+j-1}(XY) \leqslant \lambda_j(X)\lambda_i(Y),\quad i+j \leqslant n+1, \tag{5}$$

$$\lambda_{i+j-n}(XY) \geqslant \lambda_j(X)\lambda_i(Y),\quad i+j \geqslant n+1. \tag{6}$$

Lemma 3[6] If $X, Y \in \mathbb{R}^{n\times n}$, $X = X^{\mathrm{T}} \geqslant 0$, $Y = Y^{\mathrm{T}} \geqslant 0$, then

$$\lambda_1(X)\mathrm{tr}(Y) \geqslant \mathrm{tr}(XY) = \mathrm{tr}(YX) \geqslant \lambda_n(X)\mathrm{tr}(Y). \tag{7}$$

Lemma 4[1] The upper bound of the eigenvalue of the solution P of UARE (2) satisfies:

$$\lambda_1(P) \leqslant K. \tag{8}$$

where

$$K = \frac{2\lambda_1(Q)}{a + (a^2 + 4\lambda_n(BB^{\mathrm{T}})\lambda_1(Q))^{1/2}}, \tag{9}$$

$$a = -\lambda_1(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}}) - \Delta\lambda_n(BB^{\mathrm{T}})\lambda_1(Q). \tag{10}$$

Lemma 5[7] If $X, Y \in \mathbb{R}^{n\times n}$, $X = X^{\mathrm{T}} > 0$, $Y = Y^{\mathrm{T}} \geqslant 0$, then

$$\mathrm{tr}((I + XY)^{-1}X) \geqslant \frac{\mathrm{tr}(X)}{1 + \lambda_1(X)\lambda_1(Y)}. \tag{11}$$

3 Main results

Theorem 1 For the trace $\mathrm{tr}(P)$ of the solution P of UARE (2), the following inequality holds:

$$\mathrm{tr}(P) \leqslant \mathrm{tr}(\Delta Q) + \frac{nK(1 + \Delta\lambda_1(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}}))}{1 + K\lambda_n(\Delta BB^{\mathrm{T}})}, \tag{12}$$

where K is given by Eq. (9) and n is the dimension of the square matrix A.

Proof By using the matrix identity

$$(X^{-1} + YZ^{-1}) = X - XY(I + ZXY)^{-1}ZX \tag{13}$$

where X, Y and Z are constant matrices with appropriate dimensions. Eq. (2) can be transformed to

$$P = (\Delta A + I)^{\mathrm{T}}(P^{-1} + \Delta BB^{\mathrm{T}})^{-1}(\Delta A + I) + \Delta Q. \tag{14}$$

Taking traces on both sides of above equality, in the light of Eqs. (4) and (7) such that

$$\begin{aligned}
&\mathrm{tr}(P) - \mathrm{tr}(\Delta Q) = \\
&\mathrm{tr}((\Delta A + I)^{\mathrm{T}}(P^{-1} + \Delta BB^{\mathrm{T}})^{-1}(\Delta A + I)) \leqslant \\
&\mathrm{tr}((\Delta A + I)(\Delta A + I)^{\mathrm{T}})\lambda_1((P^{-1} + \Delta BB^{\mathrm{T}})^{-1}) = \\
&\frac{\mathrm{tr}((\Delta A + I)(\Delta A + I)^{\mathrm{T}})}{\lambda_n(P^{-1} + \Delta BB^{\mathrm{T}})} \leqslant \\
&\frac{\mathrm{tr}((\Delta A + I)(\Delta A + I)^{\mathrm{T}})}{\lambda_n(P^{-1}) + \lambda_n(\Delta BB^{\mathrm{T}})} = \\
&\frac{\mathrm{tr}((\Delta A + I)(\Delta A + I)^{\mathrm{T}})\lambda_1(P)}{1 + \lambda_1(P)\lambda_n(\Delta BB^{\mathrm{T}})}.
\end{aligned} \tag{15}$$

Since

$$\mathrm{tr}((\Delta A + I)(\Delta A + I)^{\mathrm{T}}) \leqslant n\lambda_1((\Delta A + I)(\Delta A + I)^{\mathrm{T}}) \tag{16}$$

and by using the fact

$$\lambda_i((\Delta A + I)(\Delta A + I)^{\mathrm{T}}) = 1 + \Delta\lambda_i(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}}). \tag{17}$$

Applying both (16) and (17) to (15), and using (8), thus (12) is verified.

Theorem 2 The trace of the positive definite matrix

P of UARE (2) has the following lower bound:

$$\mathrm{tr}(P) \geqslant \mathrm{tr}(Q)\left(\frac{1 + K\lambda_1(\Delta BB^{\mathrm{T}})}{K\lambda_1(BB^{\mathrm{T}}) - \lambda_n(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}})}\right) \tag{18}$$

where K is defined as Eq.(9), and $K\lambda_1(BB^{\mathrm{T}}) - \lambda_n(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}}) > 0$.

Proof Taking traces to both sides of Eq.(14), by using Eqs.(3) and (7), we obtain

$$\begin{aligned}&\mathrm{tr}(P) - \mathrm{tr}(\Delta Q) = \\ &\mathrm{tr}((\Delta A + I)^{\mathrm{T}}(P^{-1} + \Delta BB^{\mathrm{T}})^{-1}(\Delta A + I)) \geqslant \\ &\lambda_n((\Delta A + I)(\Delta A + I)^{\mathrm{T}})\mathrm{tr}((P^{-1} + \Delta BB^{\mathrm{T}})^{-1}) = \\ &\lambda_n((\Delta A + I)(\Delta A + I)^{\mathrm{T}})\mathrm{tr}((I + P\Delta BB^{\mathrm{T}})^{-1}P) \geqslant \\ &\frac{\lambda_n((\Delta A + I)(\Delta A + I)^{\mathrm{T}})\mathrm{tr}(P)}{1 + \lambda_1(P)\lambda_1(\Delta BB^{\mathrm{T}})} \geqslant \\ &\frac{\lambda_n((\Delta A + I)(\Delta A + I)^{\mathrm{T}})\mathrm{tr}(P)}{1 + K\lambda_1(\Delta BB^{\mathrm{T}})}.\end{aligned} \tag{19}$$

From (19) it follows that

$$\begin{aligned}&\mathrm{tr}(P)(1 + K\lambda_1(\Delta BB^{\mathrm{T}}) - \\ &\lambda_n((\Delta A + I)(\Delta A + I)^{\mathrm{T}})) \geqslant \\ &\mathrm{tr}(\Delta Q)(1 + K\lambda_1(\Delta BB^{\mathrm{T}})).\end{aligned} \tag{20}$$

Substituting (17) into (20) yields

$$\begin{aligned}&\mathrm{tr}(P)(K\lambda_1(\Delta BB^{\mathrm{T}}) - \Delta\lambda_n(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}})) \geqslant \\ &\mathrm{tr}(\Delta Q)(1 + K\lambda_1(\Delta BB^{\mathrm{T}})),\end{aligned} \tag{21}$$

when $K\lambda_1(BB^{\mathrm{T}}) - \lambda_n(A + A^{\mathrm{T}} + \Delta AA^{\mathrm{T}}) > 0$, (18) follows from (21). This completes the proof.

Theorem 3 Let the positive definite matrix P satisfy UARE (2), then P has the upper matrix bound:

$$P \leqslant \frac{K(\Delta A + I)^{\mathrm{T}}(\Delta A + I)}{1 + K\lambda_n(\Delta BB^{\mathrm{T}})} + \Delta Q \tag{22}$$

where K is defined by Eq.(9).

Proof Applying Lemma 1 and lemma 4, we have

$$\begin{aligned}&\lambda_1((P^{-1} + \Delta BB^{\mathrm{T}})^{-1} = \\ &\frac{1}{\lambda_n(P^{-1} + \Delta BB^{\mathrm{T}})} \leqslant \frac{1}{\lambda_n(P^{-1}) + \lambda_n(\Delta BB^{\mathrm{T}})} = \\ &\frac{\lambda_1(P)}{1 + \lambda_1(P)\lambda_n(\Delta BB^{\mathrm{T}})} \leqslant \frac{K}{1 + K\lambda_n(\Delta BB^{\mathrm{T}})}.\end{aligned} \tag{23}$$

Note that the following inequality holds

$$(P^{-1} + \Delta BB^{\mathrm{T}})^{-1} \leqslant \lambda_1((P^{-1} + \Delta BB^{\mathrm{T}})^{-1})I_n. \tag{24}$$

Substituting (24) into (14) gives (22). This ends the proof.

Remark 2 1) For $\Delta = 0$, (12)(18) and (22) tends to new bounds for CARE, respectively.

2) For $\Delta = 1$ and using Remark 1, the results of Theorem 1 and Theorem 2 reduce to the bounds for DARE presented in [7]; and the result of Theorem 3 is simplified to those given in [8].

3) Comparisons on tightness of bounds between obtained results and existing bounds see Ref.[7].

4 Conclusion

New trace bounds and upper matrix bound for the solution of the unified algebraic Riccati equation are obtained. Further study will be focused on bounds for the solutions of the unified non-stationary Riccati equation and the unified non-standard Riccati equation.

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