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A Staged Transformation Pseudolinearization Method for a Class of Nonlinear System *

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Abstract: For single input nonlinear systems, this paper gives a staged transformation pseudolinearization method and its algorithm. Firstly the original system is transformed into mormal form. Then making use of the algorithm in [1], the transformation T that changes normal form into psendo-normal form is obtained. Under the first transformation, parts of the states of the original system have been exact linearization and the staged transformation algorithm is easier than transforming original system into pseudolinearization directly. The computer simulation results of single inverted pendulum verify the validity of the proposed approach.

Key words: nonlinear system; pseudolinearization; inverted pendulum control
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一类非线性系统的分步变换伪线性化方法

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摘要: 给出了单输入非线性系统的分步变换伪线性化方法及其算法.首先将原系统变换为标准型.利用[1]中的算法,可以获得将标准型变换为伪标准型的变换阵 T.在第一步变换下,原系统的部分状态已经获得精确线性化,并且分步变换算法比由原系统直接伪线性化的算法简单.单杆倒立摆的计算机仿真表明了该算法的有效性.

关键词: 非线性系统; 伪线性化; 倒立摆控制

1 Introduction

Over the last decade, researchers have been investigating various procedures in linearizing nonlinear systems - including local linearization, operating set linearization and exact linearization. The effective region of local linearization is very limited. The exact input-state linearization[2] transformation is difficult to acquire and sometimes does not exist. Furthermore when zero-dynamics are unstable, the method of exact input-output linearization is difficult to use, though the transformations is easier to obtain.

Extended-linearization introduced by Rugh and Baumann[3,4] is a design method based on the family of linearization of the system, parameterized by the family of operating point set. Teboulet and Champetier[1,5] considered another operating point set linearization method: pseudolinearization, i.e. via state feedback and state coordinate change such that, in the new coordinate the linearization model is independent of the operating point set. Lawrence and Rugh[6,7] had developed the method into an input-output version.

This paper proposes a staged transformation pseudolinearization method for general single input nonlinear systems. Firstly, we get the normal form, which is partly exact linearization. Then, we obtain a specific algorithm, which transforms normal form into pseudo-normal form. Finally, we give the asymptotic stable control law and the computer simulation results of single inverted pendulum.

2 Staged transformation algorithm

Consider a single input-output nonlinear system

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$$\begin{cases} \dot{x} = f(x,u), \\ y = h(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^1$, is an input, $f(\cdot,\cdot):\mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ with $f(0,0) = 0$ and $y \in \mathbb{R}^1$, is an output.

2.1 Normal form

Resembling the algorithm of [8], differentiate the output $y(t)$ in series until

$$\partial y^{(i)}(x,u)/\partial u \neq 0.$$

Set $r = i$ and $y^{(r)}(x,u) = v$. The number r is named relative order of Eq. (1) and we can obtain the first staged state feedback

$$u = y^{-1_{(r)}}(x,v). \tag{2}$$

We can find[1]

$$z_J = [\eta_1 \cdots \eta_{n-r}], \tag{3}$$

such that $\partial[y, y^1 \cdots y^{r-1}, \eta_1 \cdots \eta_{n-r}]/\partial x$ is nonsingular in some neighborhood of equilibrium point.

Setting

$$\begin{cases} z_i = y^{(i+1)},\ i = 1,2,\cdots,r, \\ z_{r+j} = \eta_j,\ j = 1,2,\cdots,n-r. \end{cases} \tag{4}$$

By state feedback (2) and state coordinate change (4), we have the new state equations in z-space

$$\begin{cases} \dot{z}_i = z_{i+1}, i = 1,2,\cdots,r-1, \\ \dot{z}_r = v, \\ \dot{z}_J = \tilde{g}(z,v). \end{cases} \tag{5}$$

Write Eq. (5) as

$$\dot{z} = g(z,v) = g(z_I, z_J, v), \tag{6}$$

where $z_I = [z_1 \quad z_2 \quad \cdots \quad z_r]$. In this paper we call Eq. (5) or Eq. (6) normal form.

Resembling the definition in [1], define operating point set of Eq. (6) as follows

$$\bar{\lambda}_{z,v} \triangleq \{(z,v), s.t.\ g(z,v) = 0\}. \tag{7}$$

Because of the special form of Eq. (5), $\bar{\lambda}_{z,v}$ can be written as

$$\bar{\lambda}_{z,v} \triangleq \{(z,v) \mid z_2 = \cdots = z_r = v = 0, \tilde{g}(z,v) = 0\}. \tag{8}$$

When $r = n$ or $r < n$ and $z_J = \tilde{g}(z,v)$ is stable in some neighborhood of the operating point set, we can synthesize system (6) making use of linear system theory. When $r < n$, $z_J = \tilde{g}(z,v)$ is unstable in some neighborhood of the operating point set, this case is the emphasis of this paper and we will descuss it in Section 2.2.

2.2 Pseudo-normal form

At first, we introduce hypotheses as follows

H1: $g(\cdot,\cdot):\mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ is analytic in a neighborhood of the operating point set and $g(0,0) = 0$.

H2: $[\partial g/\partial z \quad \partial g/\partial v] \mid \bar{\lambda}_{z,v}$ is controllable and none of the eigenvalues of $\partial \tilde{g}/\partial z_J \mid \bar{\lambda}_{z,v}$ is zero.

Otherwise, due to

$$[\partial g/\partial z \quad \partial g/\partial v] \mid \bar{\lambda}_{z,v}$$

is controllable, we can apply state feedback such that none of the eigenvalues of $[\partial \tilde{g}/\partial z_J] \mid \bar{\lambda}_{z,v}$ is zero. Now linearization of the system (6) in the neighborhood of the operating point set yields the linear state equation

$$\begin{cases} \delta\dot{z}_i = \delta z_{i+1}, i = 1,2,\cdots,r-1, \\ \delta\dot{z}_r = \delta v, \\ \delta\dot{z}_J = \delta\tilde{g}(z,v). \end{cases}$$

For convenience we simply write the above equations as

$$\delta\dot{z} = F\delta_z + G\delta_v, \tag{9}$$

where $F = \frac{\partial g}{\partial z} \mid \bar{\lambda}_{z,v}$, $G = \frac{\partial g}{\partial v} \mid \bar{\lambda}_{z,v}$.

Our aim is to find mappings

$$\begin{cases} \xi_i = T_i(z), i = 1,\cdots,n, \\ w = T_{n+1}(z,v). \end{cases} \tag{10}$$

($T_1,\cdots,T_n$ being functionally independent and $\partial T_{n+1}/\partial v \neq 0$) such that Eq. (9) in the neighborhood of $\bar{\lambda}_{z,v}$ has the form as follows

$$\begin{cases} \delta\dot{\xi}_i = \delta\xi_{i+1}, i = 1,\cdots,n-1, \\ \delta\dot{\xi}_n = \delta w. \end{cases}$$

Write the above equation as

$$\delta\dot{\xi} = A_0\delta\xi + b_0\delta w. \tag{11}$$

It is obvious that the linearization model (11) is independent of the operating points.

We call Eq. (11) pseudo-normal form. And using the algorithm of [1] we can acquire transformation (10).

2.3 Pseudo-normal form transformation

Because of the particularity of Eq. (6), the algorithm of transformation (10) is simple. In the case of H1 and H2, $\bar{\lambda}_{z,v}$ can be written as

$$\bar{\lambda}_{z,v} = \{(z,v) \mid z_2 = \cdots = z_r = v = 0, z_J = p(z_I)\},$$

where $z_J = p(z_I)$ is decided by

$$\tilde{g}(z_I, z_J, v) \mid \bar{\lambda}_{z,v} = 0$$

and

$$p(z_I) = p(z_1) = [p_1(z_1) \quad p_2(z_1) \quad \cdots \quad p_{n-r}(z_1)].$$

So the 1-form of transformation T_i, $i = 1, \cdots, n+1$ can be chosen as

$$\begin{cases} \mathrm{d}T_i \mid \bar{\lambda}_z = \alpha_i(z_1), \\ \mathrm{d}T_{n+1} \mid \bar{\lambda}_{z,v} = \alpha_{n+1}(z_1), \end{cases} \tag{12}$$

where

$$\alpha_i(z_1) = [\alpha_{i,1}(z_1)\ \alpha_{i,2}(z_1)\ \cdots\ \alpha_{i,n}(z_1)],$$
$$i = 1, 2, \cdots, n,$$
$$\alpha_{n+1} = [\alpha_{n+1,1}(z_1)\ \alpha_{n+1,2}(z_1)\ \cdots\ \alpha_{n+1,n+1}(z_1)],$$

$\alpha_i(i = 1, 2, \cdots, n+1)$ can be calculated by the formulas in [1].

2.4 Transformation algorithm

The transformation (10) can be written as

$$\begin{cases} T_i = \phi_i(z_1) + \sum_{j=2}^{n} \alpha_{ij} z_j, \\ T_{n+1} = \phi_{n+1}(z_1) + \sum_{j=2}^{n} \alpha_{n+1,j} z_j + \alpha_{n+1,n+1} v. \end{cases}$$

From Eq. (12) $\phi_i(z_1)(i = 1, 2, \cdots, n+1)$ can be expressed by

$$\phi_i(z_1) = \int (\alpha_{i,1}(z_1) - \sum_{j=r+1}^{n} p_{j-r}(z_1) \frac{\mathrm{d}}{\mathrm{d}z_1} \alpha_{i,j}) \mathrm{d}z_1,$$
$$i = 1, 2, \cdots, n,$$

and

$$\phi_{n+1}(z_1) = \int (\alpha_{n+1,1}(z_1) - \sum_{j=r+1}^{n} p_{j-r}(z_1) \frac{\mathrm{d}}{\mathrm{d}z_1} \alpha_{n+1,j}) \mathrm{d}z_1.$$

Because of $z_2 = 0, \cdots, z_r = v = 0$ in $\bar{\lambda}_{z,v}$, compared with the algorithm in [1] the T_i, $i = 1, \cdots, n+1$ are easier to be obtained. And the larger the number r is, the easier the transformations are to be acquired until $r = n$ the transformation $T = T(z)$ is equal to $I_{n \times n}$.

2.5 The special case

Now we consider a special case i.e. when Eq. (6) satisfies condition C1, how to obtain transformations T_i, $i = 1, \cdots, n+1$.

C1: $\partial \bar{g} / \partial z_1 \equiv 0$.

Then we have a theorem as follows:

Theorem 1 If Eq. (6) satisfies condition C1, T_i, $i = 1, \cdots, n$ are linear transformations and state feedback $v = w - qz$, where $q = [q_1 \quad q_2 \quad \cdots \quad q_n]$ is constant row vector.

Proof Under condition C1 and hypotheses H1 and H2, $\bar{\lambda}_{z,v}$ can be written as

$$\bar{\lambda}_{z,v} = \{(z, v) \mid z_i = 0, i = 2, 3, \cdots, n, v = 0\},$$

thus Eq. (9) can be expressed as

$$\delta \dot{z} = A \delta z + b \delta v, \tag{13}$$

where A, b are constant matrix. Assume $\delta \xi = T \delta z$ and $\delta v = \delta w - q \delta z$ are the required change such that Eq. (11) is satisfied, where q is the last line of TAT^{-1}. From Section 2.4, we know that $v = w - qz$ is a solution of $\delta v = \delta w - q \delta z$. Under the transformation $\xi = Tz$, Eq. (6) can be written as

$$T^{-1} \dot{\xi} = g(T^{-1} \xi, v). \tag{14}$$

Linearizing Eq. (14) in the neighborhood of $\bar{\lambda}_{\xi,w}$ yields

$$T^{-1} \delta \dot{\xi} = AT^{-1} \delta \xi + b \delta v,$$

substitute δv with $\delta w - q \delta z$, then we can obtain the form of Eq. (11). Q.E.D.

2.6 Asymptotic stable control law

In order to obtain closed asymptotic stable system in the neighborhood of the operating point set, we consider the state feedback control laws of Eq. (11) has the form

$$w = -k(\xi - \xi^*), \tag{15}$$

where ξ^* is the equilibrium point and k may be decided by LQR optimal control.

Theorem 2 Suppose that system (6) can be transformed into Eq. (11), then the control law

$$u = h_r^{-1}(x, T_{n+1}^{-1}[z(x), -k(\xi(z(x)) - \xi^*)]) \tag{16}$$

can stabilize original system (1) in the neighborhood of the operating point set.

Proof The control law can be easily obtained from state coordinate change (4), (10) and state feedback (2), (15). Q.E.D.

3 Some analyses

3.1 System differentiability

We do not suppose that the right hand of the state equation (1) is sufficiently differentiable, i.e. $f(x, u)$ is sufficiently differentiable in the neighborhood of the operating point set. The property makes the algorithm even be able to deal with some system with nondifferentiable parts. The state equations of inverted pendulum in Section 4 are an example. Making use of the property, we can deal with some more general nonlinear system.

3.2 Part exact linearization

The difference between transforming Eq. (1) to Eq. (11) and transforming Eq. (6) to Eq. (11) is that Eq. (6) is partly exact linearization which weakens the nonlinearity of original system.

3.3 Approximate optimization

For single input affine nonlinear system, Tan[8] has proved that optimal control law of pseudolinearization system is the approximate optimal control law of the original nonlinear system.

Suppose Eq. (1) is an affine nonlinear system. In Eq. (15), let $k = bP$, where P is the unique positive-definite solution of the algebraic Riccati equation

$$PA_0 + A_0^{\mathrm{T}}P - Pb_0b_0^{\mathrm{T}}P + Q = 0,$$

and Q is a symmetric, positive-definite matrix.

From [8], we know that feedback (15) is the approximate optimal control of Eq. (6) with performance index

$$J = \frac{1}{2}\int_0^{\infty}\{T^{\mathrm{T}}(z)QT(z) + T_{n+1}^2(z,v)\}\mathrm{d}t.$$

From Eq. (2) and Eq. (4), we can write performance index as

$$J = \frac{1}{2}\int_0^{\infty}\{T^{\mathrm{T}}(z(x))QT(z(x)) + T_{n+1}^2(z(x),v(x,u))\}\mathrm{d}t.$$

So control law (15) is an approximate nonlinear optimal law of original system (1), too.

4 Single inverted pendulum control

To illustrate the design method, consider the problem of balancing an inverted pendulum on a cart[9].

Friction considered, the inverted pendulum is described by the following differential equations:

$$\begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = \dfrac{3}{4l}(g\sin x_1 - a\cos x_1 - \dfrac{1}{m_pl}\mu_p x_2),\\ \dot{x}_3 = x_4,\\ \dot{x}_4 = a_F + a_1 - a_2 - \dfrac{1}{m}\mu_c\mathrm{sgn}(x_4),\end{cases} \tag{17}$$

where x_1 is the angle (in radians) of the pendulum from vertical, and x_3 is the displacement of the cart from the origin, and F is the force as control applied to the cart (in Newtons). $a = \dot{x}_4$, $a_1 = (m_p/m)lx_2^2\sin x_1$, $a_2 = (m_p/m)l\dot{x}_2\cos x_1$, $a_F = (1/m)F$ and $m = m_c + m_p$. The various parameters and the values used in the simulation see [9].

In order to obtain normal form, we choose $y = x_3$. It is straightforward to write the state coordinate change as

$$z_1 = x_3,\ z_2 = x_4,\ z_3 = x_1,\ z_4 = x_2,$$

and state feedback

$$F = v \times (m - \frac{3m_p}{4l}\cos(z_3)) + \frac{3m_p}{4l}(g\ \sin z_3 - \frac{1}{m_pl}\mu_p z_4) + \mu_c\mathrm{sgn}(z_2).$$

In z-space, the state equations have the form

$$\begin{cases}\dot{z}_1 = z_2,\\ \dot{z}_2 = v,\\ \dot{z}_3 = z_4,\\ \dot{z}_4 = \dfrac{3}{4l}(g\ \sin z_3 - v\ \cos z_3 - \dfrac{1}{m_pl\mu_pz_4}).\end{cases} \tag{18}$$

It is easy to verify that Eq. (18) has an unstable zero dynamics, i.e. it is nonminimum phase system, so we can not synthesize it by exact feedback linearization[1].

It is obvious that Eq. (18) satisfies condition C1, so the transformation T is a constant matrix.

Linerizing Eq. (18) in the neighborhood of $\bar{\lambda}_{z,v}$ yields

$$\delta z = A\delta z + b\delta V. \tag{19}$$

From Theorem 1, we can obtain $\xi = Tz$ and in the end we can obtain the form of Eq. (11).

Choose $Q = [1\ \ 0\ \ 1\ \ 0]$, $R = 1$ and solve the Riccati equation

$$PA + A^{\mathrm{T}}P - Pbb^{\mathrm{T}}P + Q = 0.$$

We can obtain LQR optimal control law

$$\delta v = v - v^* = -k\delta z,$$

where $-k = -(Pb)' = [1\ \ 1.9\ \ 2.8\ \ 7.4]$.

Based on Theorem 2, we can obtain the final control law as follows

$$\begin{aligned}F = {} & (x_3 + 7.4x_4 + 28x_1 + 1.9x_2) \times \\ & (1.1 - \frac{3}{40}\cos^2 x_1) + \\ & \frac{3}{40}(9.8\sin x_1 - 0.00004x_2) \times \cos x_2 + \\ & 0.0005\mathrm{sign}(x_4) - 0.05x_2^2\sin x_1.\end{aligned} \tag{20}$$

The responses of the system with control law (20) are shown for four values of $x_1(0)$ with $x_2(0) = x_3(0) = x_4(0) = 0$ in a, b of Fig. 1 as curves 1, 2, 3, 4 respectively.

The responses of the system with control law (20) are shown for four values of $x_3(0)$ with $x_2(0) = x_4(0) = 0$, $x_1 = 0.5236 = 30°$ in c, d of Fig. 1 as curves 1, 2, 3, 4 respectively.

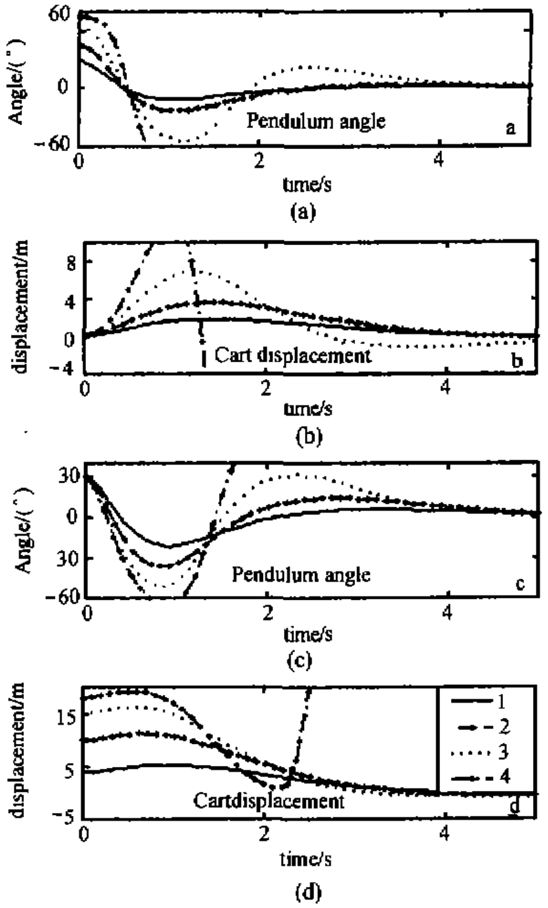

Fig. 1 Response curves

5 Conclusion

This paper gives a staged transformation pseudolinearization method. Firstly the original system is transformed into normal form, then the normal form is transformed into pseudo-normal. In this paper, we give the staged transformation algorithm. By staged transformation, exact linearization of part states is obtained and the staged transformation algorithm is an easier algorithm than transforming original system into pseudo-linearization directly. In Section 4, the results of computer simulation of single inverted pendulum verify the validity of the proposed approach.

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