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Stabilization of nonhomogeneous beam by embedding patch of Kelvin-Voigt viscoelasticity

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Abstract: With the help of operator semigroup theory, the frequency domain method and the multiplier techniques were applied. When the Kelvin-Voigt damping was distributed locally on any subinterval of the region, the energy of the nonhomogeneous Euler-Bernoulli beam was proved to decay uniformly exponentially.

Key words: Euler-Bernoulli beam equation; local Kelvin-Voigt damping; multiplier; exponential stability

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嵌有 Kelvin-Voigt 阻尼片的非均质梁的镇定

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摘要: 借助于算子半群理论.应用频域方法和乘子技巧,证明了具有局部 Kelvin-Voigt 阻尼的非均质 Euler-Bernoulli 梁的能量是指数衰减的.

关键词: Euler-Bernoulli 梁方程; 局部 Kelvin-Voigt 阻尼; 乘子; 指数稳定

1 Introduction

Embedding viscoelastic patches in an elastic structure is a stabilization technique in engineering. Consider a clamped nonhomogeneous elastic beam of length L. One segment of the beam is made of a viscoelastic material with the Kelvin-Voigt constitutive relation. By employing the Kirchhoff hypothesis and neglecting the rotatory effect, the transverse vibration of the beam can be described as the following Euler-Bernoulli beam equation with the Kelvin-Voigt damping and initial-boundary conditions,

$$\begin{cases} \rho w_{tt} + (pw'' + D_b w_t'')'' = 0 \text{ in } (0,L) \times \mathbb{R}^+, \\ w(0,t) = w(L,t) = w'(0,t) = w'(L,t) = 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ x \in (0,L), \end{cases} \tag{1.1}$$

where the prime represents the derivative with respect to the spacial variables $x, w, L, \rho(x) > 0, p \equiv EI(x) > 0$ are the transversal displacement, length, density, and flexural rigidity modulus of the beam, and $D_b \geqslant 0$ is the damping coefficient which is strictly positive on a proper subinterval $[\alpha,\beta]$ of $[0,L]$ but vanishes on $[0,L] \setminus [\alpha,\beta]$. The energy of solution to (1.1) at time t is

$$E(t) = \frac{1}{2}\int_0^1 [p(x) \mid w''(x,t) \mid^2 + \rho(x) \mid w_t(x,t) \mid^2] \mathrm{d}x. \tag{1.2}$$

The purpose of this paper is to prove that the energy of the beam decays uniformly exponentially, i. e.,

$$E(t) \leqslant M \mathrm{e}^{-\mu t} E(0),\ t \geqslant 0 \tag{1.3}$$

for some $\mu > 0, M \geqslant 1$ and all initial values of finite energy. When ρ and p are constants on $[0,\alpha] \cup [\beta,L]$, Liu K and Liu Z[1] proved the energy decay property (1.3) by the frequency domain method. We assume

Assumption A

i) $0 < \rho \in C^1[0,\alpha] \cup C^2[\alpha,\beta] \cup C^1[\beta,L]$;

ii) $D_b = 0$ on $[0,\alpha) \cup (\beta,L], 0 < D_b \in C[\alpha,\beta]$;

iii) $0 < p \in C^2[0,\alpha] \cup C[\alpha,\beta] \cup C^2[\beta,L]$.

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We will prove

Theorem 1 Under Assumption A on the coefficients, the energy $E(t)$ satisfies the uniform exponential decay property (1.3).

Recently, Liu K and Liu Z introduced two efficient multipliers to establish the boundary exponential stabilization of the nonhomogeneous beam equation with only bending moment feedback in the preprint. In this paper we will employ the similar multipliers and the frequency domain method to prove our theorem.

2 Proof of Theorem 1

Let $H = L^2_\rho(0,L)$ with the norm $\| v \| = (\int_0^L \rho | v(x) |^2 dx)^{\frac{1}{2}}$ and $V = H_0^2(0,L)$ with the norm $\| v \|_V = (\int_0^L p | v''(x) |^2 dx)^{\frac{1}{2}}$. Define $\mathscr{H} = V \times H$ with the norm $\| (w,v) \|_{\mathscr{H}} = (\| w \|_V^2 + \| v \|^2)^{\frac{1}{2}}$. Then $\mathscr{H}$ is a Hilbert space — the finite energy state space. Define in $\mathscr{H}$ that

$$D(\mathscr{A}) = \{(w,v) | w, v \in V, -M \equiv pw'' + D_b v'' \in H^2(0,L)\}, \tag{2.1}$$

and

$$\mathscr{A}(w,v) = (v, \frac{1}{\rho} M''). \tag{2.2}$$

Then, the equation (1.1) can be written as the following abstract evolution equation on $\mathscr{H}$,

$$\begin{aligned} (\dot{w}(t), \dot{v}(t)) &= \mathscr{A}(w(t), v(t)), \\ (w(0), v(0)) &= (w_0, w_1). \end{aligned} \tag{2.3}$$

It is known that $\mathscr{A}$ generates a C_0-semigroup, $e^{t\mathscr{A}}$, of contractions on $\mathscr{H}$ (see Liu K and Liu Z[1]). Therefore, $(w(t), \dot{w}(t)) = e^{\mathscr{A}t}(w_0, w_1)$ gives the mild solution of (1.1) for every $(w_0, w_1) \in \mathscr{H}$. Moreover, $\mathscr{A}^{-1}$ is a bounded operator on $\mathscr{H}$. It is clear that the exponential decay property (1.3) is equivalent to the exponential stability of the C_0-semigroup $e^{t\mathscr{A}}$.

Lemma 1 $R(\mathscr{A}) \subset \rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$. **Proof of Theorem 1** By the frequency domain condition for the exponential stability of C_0-semigroups on Hilbert spaces[2], we only need to prove that

$$\sup \{ \| (\lambda - \mathscr{A})^{-1} \| \ \lambda \in R(\mathscr{A}) \} < +\infty. \tag{2.4}$$

Suppose (2.4) is not true, by the continuity of the resolvent and the Resonance theorem, there exist $\lambda_n \in R(\mathscr{A})$, $(w_n, v_n) \in D(\mathscr{A})$, $n = 1,2,\cdots$, such that

$$\| (w_n, v_n) \|_{\mathscr{H}} = 1, \ | \lambda_n | \to \infty, \tag{2.5}$$

and

$$(\lambda_n - \mathscr{A})(w_n, v_n) \equiv (f_n, g_n) \to 0 \text{ in } \mathscr{H}. \tag{2.6}$$

Equation (2.6) implies

$$\lambda_n w_n - v_n = f_n \to 0 \text{ in } V, \tag{2.7}$$

$$\lambda_n v_n \rho - M_n'' = \rho g_n \to 0 \text{ in } L^2(0,L), \tag{2.8}$$

where $M_n = -(pw_n'' + D_b v_n'')$. From (2.6), we have

$$\mathrm{Re} \ \langle (\lambda_n - \mathscr{A})(w_n, v_n), (w_n, v_n) \rangle_{\mathscr{H}} = \int_\alpha^\beta D_b | v_n'' |^2 dx \to 0. \tag{2.9}$$

We substitute (2.7) into (2.8) to get

$$\lambda_n^2 \rho w_n - M_n'' = \rho(g_n + \lambda_n f_n) \text{ for } x \in (0,L). \tag{2.10}$$

Multiply the above equation by $\bar{w}_n$, then integrate by parts on $(0,L)$. This leads to

$$\| \lambda_n w_n \|^2 - \| w_n \|_V^2 \to 0. \tag{2.11}$$

Here, we have used (2.5), (2.7), (2.8) and (2.9).

The rest of the proof depends on the following lemmas. Let $\theta_n = \sqrt{| \lambda_n |}$.

Lemma 2 (Liu K and Liu Z[1]) The sequence $\{w_n\}$ of functions has the following properties:

$$\lambda_n w_n \to 0 \text{ in } H^2(\alpha, \beta), \tag{2.12}$$

$$\theta_n^4 (| w_n(\alpha) |^2 + | w_n'(\alpha) |^2 + | w_n(\beta) |^2 + | w_n'(\beta) |^2) \to 0, \tag{2.13}$$

$$\theta_n^{-1} w_n''(\alpha^-), \ \theta_n^{-1} w_n''(\beta^+) \to 0, \tag{2.14}$$

$$\theta_n^{-1} w_n'''(\alpha^-), \ \theta_n^{-1} w_n'''(\beta^+) \to 0, \tag{2.15}$$

On the intervals $(0,\alpha)$ and (β, L), equation (2.10) becomes

$$\lambda_n^2 \rho w_n + (p w_n'')'' = \rho(g_n + \lambda_n f_n). \tag{2.16}$$

Lemma 3

$$w_n''(\alpha^-) \to 0 \text{ as } n \to \infty, \tag{2.17}$$

$$w_n''(\beta^+) \to 0 \text{ as } n \to \infty. \tag{2.18}$$

Since $\| w_n \|_V^2 + \| v_n \|^2 = 1$ and $\lambda_n w_n - v_n$ also converges to zero in $L^2(0,L)$, (2.11) implies that both $\| \lambda_n w_n \|^2$ and $\| w_n \|_V^2$ must converge to $\frac{1}{2}$ as $n \to \infty$. This further leads to

$$\lim_{n \to \infty} (\int_0^\alpha + \int_\beta^L) \rho | \lambda_n w_n |^2 dx = \lim_{n \to \infty} (\int_0^\alpha + \int_\beta^L) p | w_n'' |^2 dx = \frac{1}{2} \tag{2.19}$$

when (2.12) is taken into account.

In what follows, we will show that

$$\lim_{n \to \infty} (\int_0^\alpha | \lambda_n w_n |^2 dx + \int_\beta^L | \lambda_n w_n |^2 dx) = 0 \tag{2.20}$$

to get a contradiction with (2.19).

Multiply equation (2.16) by $\psi\overline{w'_n}$ (where $\psi = e^{\eta x} - 1$, η is a positive constant which will be determined later on) and integrate on $(0,\alpha)$, and then take the real part, we obtain

$$\mathrm{Re}\int_0^\alpha \lambda_n^2 \rho w_n \psi \overline{w'_n}\mathrm{d}x + \mathrm{Re}\int_0^\alpha (pw'')''\psi\overline{w'_n}\mathrm{d}x =$$

$$\mathrm{Re}\int_0^\alpha \rho(g_n + \lambda_n f_n)\psi\overline{w'_n}\mathrm{d}x. \quad (2.21)$$

The right-hand side of (2.21) converges to zero. After a straightforward calculation (integration by parts), we have

$$\mathrm{Re}\int_0^\alpha \lambda_n^2 \rho w_n \psi \overline{w'_n}\mathrm{d}x =$$

$$\frac{1}{2}\int_0^\alpha (\rho'\psi + \rho\psi')\,|\lambda_n w_n|^2\mathrm{d}x + o(1), \quad (2.22)$$

$$\mathrm{Re}\int_0^\alpha (pw''_n)''\psi\overline{w'_n}\mathrm{d}x =$$

$$-\mathrm{Re}\left\{\int_0^\alpha (pw''_n)'\psi'\overline{w'_n}\mathrm{d}x + \int_0^\alpha (pw''_n)'\psi\overline{w''_n}\mathrm{d}x\right\} + o(1), \quad (2.23)$$

$$\mathrm{Re}\int_0^\alpha (pw''_n)'\psi'\overline{w'_n}\mathrm{d}x =$$

$$-\int_0^\alpha p\psi'\,|w''_n|^2\mathrm{d}x + o(1), \quad (2.24)$$

$$\mathrm{Re}\int_0^\alpha (pw''_n)'\psi\overline{w''_n}\mathrm{d}x =$$

$$\mathrm{Re}\left\{\int_0^\alpha p'\psi\,|w''_n|^2\mathrm{d}x + \int_0^\alpha pw'''_n\psi\overline{w''_n}\mathrm{d}x\right\}, \quad (2.25)$$

$$\mathrm{Re}\int_0^\alpha pw'''_n\psi\overline{w''_n}\mathrm{d}x =$$

$$\mathrm{Re}\left\{(p\psi|w''_n|^2)|_0^\alpha - \int_0^\alpha [(p'\psi + p\psi')|w''_n|^2 + pw''_n\psi\overline{w'''_n}]\mathrm{d}x\right\}. \quad (2.26)$$

From (2.17) and (2.26), we have

$$\mathrm{Re}\int_0^\alpha pw'''_n\psi\overline{w''_n}\mathrm{d}x =$$

$$-\frac{1}{2}\int_0^\alpha (p'\psi + p\psi')\,|w''_n|^2\mathrm{d}x + o(1). \quad (2.27)$$

From (2.25) and (2.27), we deduce

$$\mathrm{Re}\int_0^\alpha (pw''_n)'\psi\overline{w''_n}\mathrm{d}x =$$

$$\frac{1}{2}\int_0^\alpha (p'\psi - p\psi')\,|w''_n|^2\mathrm{d}x + o(1). \quad (2.28)$$

Substituting (2.24), (2.28) into (2.23), we have

$$\mathrm{Re}\int_0^\alpha (pw''_n)''\psi\overline{w'_n}\mathrm{d}x =$$

$$\int_0^\alpha \left(\frac{3}{2}p\psi' - \frac{1}{2}p'\psi\right)|w''_n|^2\mathrm{d}x + o(1). \quad (2.29)$$

(2.21), (2.22) and (2.29) imply that

$$\int_0^\alpha \frac{1}{2}(\rho'\psi + \rho\psi')\,|\lambda_n w_n|^2\mathrm{d}x +$$

$$\int_0^\alpha \left(\frac{3}{2}p\psi' - \frac{1}{2}p'\psi\right)|w''_n|^2\mathrm{d}x = o(1). \quad (2.30)$$

If η is taken big enough, we can make

$$\frac{1}{2}(\rho'\psi + \rho\psi') > c_1,\ \frac{3}{2}p\psi' - \frac{1}{2}p'\psi > c_1,$$

here c_1 is a positive constant. Thus we have proved

$$\lim_{n\to\infty}\left(\int_0^\alpha |\lambda_n w_n|^2\mathrm{d}x + \int_0^\alpha |w_n''|^2\mathrm{d}x\right) = 0. \quad (2.31)$$

Multiply equation (2.16) by $(e^{\eta(L-x)} - 1)\overline{w'_n}$ and integrate on (β, L), where η is a positive constant large enough. Similarly, we can prove

$$\lim_{n\to\infty}\left(\int_\beta^L |\lambda_n w_n|^2\mathrm{d}x + \int_\beta^L |w''_n|^2\mathrm{d}x\right) = 0. \quad (2.32)$$

Thus (2.20) follows from (2.31) and (2.32).

So far, we have got the promised contradiction.

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