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Mixed H_2/H_∞ filtering with regional pole assignment

YANG Fu-wen

(Department of Electrical Engineering, Fuzhou University, Fujian Fuzhou 350002, China)

Abstract: The mixed H_2/H_∞ filtering problem with regional pole assignment was addressed for linear continuous time-invariant systems. A general framework for solving this problem was established using LMI approach in conjunction with regional stability constraints, H_2 and H_∞ optimization characterizations. The necessary and sufficient conditions for the solvability of the problem were given in terms of a set of LMI's. A numerical example was provided to illustrate the proposed design approach.

Key words: mixed H_2/H_∞ filtering; regional pole assignment; LMI

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具有域极点配置的混合 H_2/H_∞ 滤波

杨富文

(福州大学 电气工程系，福建 福州 350002)

摘要：解决了具有域极点配置的连续时不变系统的混合 H_2/H_∞ 滤波问题.通过采用线性矩阵不等式(LMI)方法描述域稳定性限制、H_2 和 H_∞ 优化，以建立求解这个问题的总框架.这个问题的可解性的充分必要条件由一组 LMI 给出.最后用一个数字例子来说明所给出的设计方法.

关键词：混合 H_2/H_∞ 滤波；域极点配置；线性矩阵不等式

1 Introduction

The state estimation of dynamic systems in the presence of both process and measurement noise poses a very important problem in the engineering applications. One celebrated design approach is the Kalman filtering (also called as H_2 filtering), which minimizes the H_2 norm of the estimation error under the assumptions of the noise processes with known power spectral densities[1]. In practice, however, the noise processes often have unknown or uncertain spectral densities. This difficulty has been overcome by reformulating the estimation problem in an H_∞ filtering framework during the last few years[2]. Moreover, de Souza et al[3] have presented an example to demonstrate that the H_∞ filtering is more robust to plant uncertainties than the H_2 filtering.

The H_∞ filtering offers a performance that is significantly better than the corresponding performance of the H_2 filtering. However, the H_∞ filtering typically leads to a large intolerable estimation error variance when the system is driven by white noise signals. The advantage of H_2 filtering is that the variance of the estimation error is minimized. The mixed H_2/H_∞ filtering approach that considers the co-presence of two sets of exogenous signals was introduced as an attempt to capture the benefits of the two filters. The mixing approach allows one to trade off between the best performance of the H_2 filter and the best guaranteed worst-case performance of the H_∞ filter. As a result, the optimal mixed H_2/H_∞ filters achieve the best performance, not over the set of all filters, but over a restricted set of filters that achieve a certain worst-case performance bound. However, unlike the H_2 and H_∞ filtering problems, with readily computable solutions, no ideal solution has been suggested to the mixture problem, even though the following approaches that have been used to address the mixed H_2/H_∞ filtering problem.

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Bernstein and Haddad[4] first transformed the mixed H_2/H_∞ filtering problem into an auxiliary minimization problem, and then, gave the solution which leads to an upper bound on the H_2 filtering error by solving a set of coupled Riccati and Lyapunov equations using Lagrange multiplier technique. A time domain game approach was proposed to solve the problem through a set of coupled Riccati equations[5,6]. Khargonekar et al[7] and Rotstein et al[8] have used convex optimization approach to obtain the solutions involving affine symmetric matrix inequalities. An alternative approach, LMI approach, will be introduced in this paper to solve this problem. Since LMI's intrinsically reflect constraints rather than optimality, they offer more flexibility for combining several constraints. LMI's can be solved by efficient interior-point optimization algorithms, such as those described in [9 ~ 11]. Moreover, software like MATLAB's LMI Toolbox is now available to solve such LMI's efficiently.

On the other hand, the standard mixed H_2/H_∞ filter design primarily concerns the stability and frequency-domain performance specifications of the filter, and provides little control over estimation dynamics. As is well known, the dynamics of a linear system is related to the location of its poles. By constraining the filter's poles to lie inside a prescribed region of the left-half complex plane, the filter designed will have a desired performance. Moreover, the locations of the filter's poles can improve the transient behavior, and also provide indirect tolerance against structured uncertainties. In this paper, we will address the mixed H_2/H_∞ filtering problem with regional pole assignment for linear continuous time-invariant systems. To our knowledge, the mixed H_2/H_∞ control problem with regional pole assignment has been considered in [12], but the mixed H_2/H_∞ filtering problem with regional pole assignment remains open. The approach to be used is different from that proposed in [12], where the Lagrange multiplier technique was used to solve a set of highly coupled Riccati and Lyapunov equations. We transform the H_2 norm bound, H_∞ norm bound and pole clustering into LMI formulation. After straightforward manipulations with the help of the variable changes, the overall problem remains convex. The solutions can be readily obtained by using existing LMI Toolbox.

The notation used here is fairly standard. $\otimes$ denotes the Kronecker product; $\|\cdot\|_p$ stands for H_p-norm in Hardy space; M^H denotes the Hermitian transpose of matrix M; tr (M) represents the trace of matrix M; $\bar{\lambda}$ denotes the conjugate of λ; the shorthand diag $\{M_1, M_2, \cdots, M_N\}$ denotes a block diagonal matrix whose diagonal blocks are given by M_1, M_2, etc.

2 Problem formulation

Consider a linear time-invariant system described by the state space model:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 v(t), \\ y(t) = Cx(t) + D_1 w(t) + D_2 v(t), \\ z_\infty(t) = L_\infty x(t), \\ z_2(t) = L_2 x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the measured output, $z_\infty(t) \in \mathbb{R}^{m1}$ represents a combination of the states to be estimated, and $z_2(t) \in \mathbb{R}^{m2}$ represents another combination of the states to be estimated. $w(t) \in \mathbb{R}^{p1}$ is a bounded power stationary disturbance input, which belongs to $L_2[0, \infty]$, and $v(t) \in \mathbb{R}^{p2}$ is a zero-mean Gaussian white noise process with unit covariance. $A, C, B_1, B_2, D_1, D_2, L_\infty$ and L_2 are known real matrices with appropriate dimensions. It will be assumed that the initial state $x(0)$ is known, and without loss of generality, we will take $x(0) = 0$. Note that when $z_\infty(t)$ and $z_2(t)$ are the same, the system (1) is reduced to those described in [5,8].

Here is the assumption:

Assumption 1 The pair (A, C) is observable.

Consider a filter for the system (1) of the form

$$\begin{cases} \dot{\hat{x}}(t) = (A - GC)\hat{x}(t) + Gy(t), \\ \hat{z}_\infty(t) = L_\infty \hat{x}(t), \\ \hat{z}_2(t) = L_2 \hat{x}(t), \end{cases} \tag{2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state, $\hat{z}_\infty(t) \in \mathbb{R}^{m1}$ is an estimate for $z_\infty(t)$, $\hat{z}_2(t) \in \mathbb{R}^{m2}$ is an estimate for $z_2(t)$, and G is filter's parameter to be determined.

Define the state estimation error

$$e(t) = x(t) - \hat{x}(t). \tag{3}$$

Then the resulting error system

$$\begin{cases} \dot{e}(t) = (A - GC)e(t) + (B_1 - GD_1)w(t) + \\ \qquad (B_2 - GD_2)v(t), \\ e_\infty(t) = z_\infty(t) - \hat{z}_\infty(t) = L_\infty e(t), \\ e_2(t) = z_2(t) - \hat{z}_2(t) = L_2 e(t). \end{cases} \tag{4}$$

Let $T_\infty(s) = L_\infty(sI - A + GC)^{-1}(B_1 - GD_1)$ be the transfer function from $w(t)$ to $e_\infty(t)$ and $T_2(s) = L_2(sI - A + GC)^{-1}(B_2 - GD_2)$ be the transfer function from $v(t)$ to $e_2(t)$. Then, the mixed H_2/H_∞ filtering problem with regional pole assignment considered in this paper is as follows.

Problem 1 (mixed H_2/H_∞ filtering problem with regional pole assignment)

Find the filter (2) which can satisfy the following design performance:

i) The H_2 criterion $\| T_2(s) \|_2$ is minimized;

ii) $T_\infty(s)$ satisfies the constraint

$$\| T_\infty(s) \|_\infty < \gamma, \tag{5}$$

where $\gamma > 0$ is a given constant; and

iii) The pole of the filter lies in the specific regions.

3 LMI formulation for H_2 norm, H_∞ norm and pole assignment

3.1 LMI formulation of H_2 norm

Assume $G(s) = C(sI - A)^{-1}B$ and A is asymptotically stable. The H_2 norm of $G(s)$ is defined by

$$\| G(s) \|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\ (G^H(j\omega)G(j\omega))\mathrm{d}\omega. \tag{6}$$

It is well known that the H_2 norm of $G(s)$ can be equivalently expressed as

$$\| G(s) \|_2^2 = \mathrm{tr}\ (CQC^T) \tag{7}$$

where $Q = Q^T$ satisfies the following Lyapunov equation

$$AQ + QA^T + BB^T = 0 \tag{8}$$

or

$$\| G(s) \|_2^2 = \mathrm{tr}\ (B^TPB) \tag{9}$$

where $P = P^T$ satisfies the following Lyapunov equation

$$A^TP + PA + C^TC = 0. \tag{10}$$

Lemma 1[13,14] (H_2 norm bound) Given any transfer function $G(s) = C(sI - A)^{-1}B$, and assume A asymptotically stable, we have

$$\| G(s) \|_2^2 < \beta \tag{11}$$

if and only if there exist symmetric positive definite matrices X and Q such that

$$\begin{bmatrix} A^TX + XA & XB \\ B^TX & -I \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} X & C^T \\ C & Q \end{bmatrix} > 0, \tag{13}$$

$$\mathrm{tr}\ (Q) < \beta. \tag{14}$$

3.2 LMI formulation of H_∞ norm

Lemma 2[10,15] (H_∞ norm bound) Given any transfer function $G(s) = C(sI - A)^{-1}B$, and assume A asymptotically stable, we have

$$\| G(s) \|_\infty < \gamma \tag{15}$$

if and only if there exists a symmetric positive definite matrix X such that

$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0. \tag{16}$$

This Lemma is known as the Bounded Real Lemma.

3.3 LMI formulation of pole assignment

Pole assignment in convex regions of the left-half plane can also be expressed as LMI constraints.

Lemma 3[16,17] (regional pole assignments) The matrix A has all its eigenvalues in the LMI region

$$D = \{\lambda \in C : f_D(\lambda) = L + \lambda M + \bar{\lambda} M^T < 0\} \tag{17}$$

where L and M are real matrices such that $L^T = L$, if and only if there exists a symmetric matrix X such that

$$M_D(A, X) = L \otimes X + M \otimes (XA) + M^T \otimes (A^TX) < 0. \tag{18}$$

According to Lemma 3, pole assignment in LMI regions can be formulated as an LMI optimization problem. In practical application, LMI regions are often specified as the intersection of elementary regions, such as vertical half-plane, vertical strips, horizontal strips, disks or conic sectors. Given LMI regions $D_1, D_2, \cdots, D_N$, the intersection

$$D = D_1 \cap D_2 \cap \cdots \cap D_N$$

has the characteristic function

$$f_D(\lambda) = \mathrm{diag}\ \{f_{D_1}(\lambda), f_{D_2}(\lambda), \cdots, f_{D_N}(\lambda)\}.$$

4 Solution to mixed H_2/H_∞ filtering problem with regional pole assignment

In the previous section, H_2 norm bound, H_∞ norm bound and pole assignment have been expressed as LMI constraints. Now we use these expressions to deal with the mixed H_2/H_∞ filtering problem with regional pole assignment. According to Lemmas 1,2 and 3, their LMI formulations are separately given.

1) The LMI formulation of $\| T_2(s) \|_2^2 < \beta$ is

$$\begin{bmatrix} (A-GC)^{\mathrm{T}}X_2+X_2(A-GC) & X_2(B_2-GD_2) \\ (B_2-GD_2)^{\mathrm{T}}X_2 & -I \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} X_2 & L_2^{\mathrm{T}} \\ L_2 & Q \end{bmatrix} > 0, \tag{20}$$

$$\mathrm{tr}\ (Q) < \beta. \tag{21}$$

2) The LMI formulation of $\| T_\infty(s) \|_\infty < \gamma$ is

$$\begin{bmatrix} (A-GC)^{\mathrm{T}}X_1+X_1(A-GC) & X_1(B_1-GD_1) & L_\infty^{\mathrm{T}} \\ (B_1-GD_1)^{\mathrm{T}}X_1 & -\gamma I & 0 \\ L_\infty & 0 & -\gamma I \end{bmatrix} < 0. \tag{22}$$

3) The LMI formulation of regional pole assignments is

$$L\otimes X_3+M\otimes(X_3(A-GC))+M^{\mathrm{T}}\otimes((A-GC)^{\mathrm{T}}X_3) < 0. \tag{23}$$

In order to recover convexity, we have to force the matrices X_1, X_2, X_3 to be the same, i.e.

$$X_1 = X_2 = X_3 = X. \tag{24}$$

This restriction is the core of the Lyapunov shaping paradigm[14]. Clearly it will bring conservatism into design. But the Lyapunov shaping paradigm offers greater flexibility than standard optimal design techniques (for detail, see [14]). Therefore, the auxiliary problem of the mixed H_2/H_∞ filtering problem with regional pole assignment which is an upper bound of the optimal H_2 performance subject to the H_∞ performance and regional pole assignment constraints is described as follows.

$$\min_{X>0,Q>0,G} \mathrm{tr}\ (Q) \text{ subject to (19),(20),(22) and (23) with}$$
$$X_1 = X_2 = X_3 = X. \tag{25}$$

The optimization problem (25) is not yet convex because it contains the products XG in LMI's (20), (22) and (23). The following theorem describes the necessary and sufficient conditions for the existence of the feasible solution to problem (25).

Theorem 1 Let D be an arbitrary LMI region contained in the open left-half plane and let (17) be its characteristic function. The problem (25) is solvable if and only if there exist symmetric positive definite matrixes X and Q, and matrix F such that the following LMIs

$$\begin{bmatrix} A^{\mathrm{T}}X-C^{\mathrm{T}}F^{\mathrm{T}}+XA-FC & XB_2-FD_2 \\ B_2^{\mathrm{T}}X-D_2^{\mathrm{T}}F^{\mathrm{T}} & -I \end{bmatrix} < 0, \tag{26}$$

$$\begin{bmatrix} -X & L_2^{\mathrm{T}} \\ L_2 & -Q \end{bmatrix} < 0, \tag{27}$$

$$\begin{bmatrix} A^{\mathrm{T}}X-C^{\mathrm{T}}F^{\mathrm{T}}+XA-FC & B_1X-FD_1 & L_\infty^{\mathrm{T}} \\ B_1^{\mathrm{T}}X-D_1^{\mathrm{T}}F^{\mathrm{T}} & -\gamma I & 0 \\ L_\infty & 0 & -\gamma I \end{bmatrix} < 0, \tag{28}$$

$$L\otimes X+M\otimes(XA-FC)+M^{\mathrm{T}}\otimes(A^{\mathrm{T}}X-C^{\mathrm{T}}F^{\mathrm{T}}) < 0 \tag{29}$$

are feasible. Moreover a suitable filter's parameter G is determined by

$$G = X^{-1}F. \tag{30}$$

Proof The results are immediately obtained by the change of the variable

$$F = XG. \tag{31}$$

According to Theorem 1, the optimization problem can be rewritten as follows:

$$\min_{X>0,Q>0,F} \mathrm{tr}\ (Q) \text{ subject to (26) - (29)}. \tag{32}$$

Remark The mixed H_2/H_∞ filtering problem with pole placement is a constrained optimization problem. So the analytic solutions are very difficult to be obtained. We have reformulated the auxiliary problem (32). Here the upper bound is used to replace the optimal H_2 performance in problem 1. Therefore the solution to (32) is not optimal mixed H_2/H_∞ filtering problem with pole placement. But our approach is more flexible, which searches all possible solutions until no specifications are met or none of the degrees of freedom are exhausted. Although it is an upper bound solution, it will be close to the optimal solution.

5 Numerical example

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -1.8 & 0.3 & -1.6 \\ 0 & -5 & 0.5 \\ 1.2 & 0.8 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} w(t) +$$

$$\begin{bmatrix} -1 \\ 0.2 \\ 0 \end{bmatrix} v(t),$$

$$y(t) = [1 \quad -0.6 \quad 2]x(t) + 0.2w(t) + 0.3v(t),$$
$$z_\infty(t) = [1 \quad 0 \quad 0.5]x(t),$$
$$z_2(t) = [1 \quad 0 \quad 2]x(t).$$

We have designed four filters for this system: H_2 filter to minimize the upper bound of $\| T_2(s) \|_2$; H_∞ filter with $\| T_\infty(s) \|_\infty < \gamma = 0.1$; H_2/H_∞ filter to minimize the upper bound of $\| T_2(s) \|_2$ subject to $\| T_\infty(s) \|_\infty < \gamma = 0.1$; H_2/H_∞ filter with regional pole assignment to minimize the upper bound of $\| T_2(s) \|_2$ subject to $\| T_\infty(s) \|_\infty < \gamma = 0.1$ and the filter's poles to be assigned inside the disk centered at $(-6,0)$ with radius 5. The true values of H_2 for different filters are given in Table 1. It can be seen from Table 1 that the true value of H_2 for H_2 filter is the smallest and the true value of H_2 for H_∞ filter is the largest. Although the optimal upper bound value for H_2/H_∞ filter with regional pole assignment is larger than that of H_2/H_∞ filter, its true value is smaller than that of H_2/H_∞ filter. The cause is that the pole constraint is applied to design the filter, and good transient property is obtained. It is evident from this example that the proposed H_2/H_∞ filter with regional pole assignment is superior hence the necessity of poles constraints.

Table 1 Comparison of results for different filters

filtering type	gain	$\| T_2(s) \|_2$	optimal upper bound
H_2/H_∞ filtering with regional pole assignment	$G = [-0.1945 \quad -3.7656 \quad 2.4183]$	0.5631	0.9179
H_2 filtering	$G = [0.5558 \quad 0.2130 \quad 1.1528]$	0.5341	0.5341
H_∞ filtering	$G = [-0.0918 \quad -8.2537 \quad 4.2777]$	0.6183	NA
H_2/H_∞ filtering	$G = [-0.2055 \quad -4.3723 \quad 2.8291]$	0.5687	0.7878

The plots of singular values for different filters are given on Fig. 1. These responses confirm that the H_∞ constraints are satisfied and the singular values stay below their respective bounds.

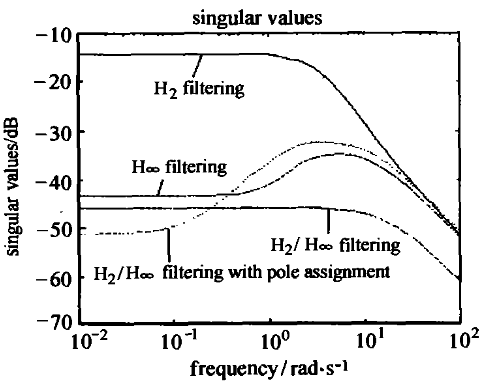

Fig. 1 Comparison of singular value responses of $T_\infty(s)$ for different filters

6 Conclusion

In this paper, we have considered the mixed H_2/H_∞ filtering problem with regional pole assignment for linear continuous time-invariant systems. An LMI approach has been developed to solve this filtering problem and the filter has been obtained in terms of a set of LMI's, which ensures not only the H_2 and H_∞ performance, but also a prescribed region of the left-half complex plane for the poles of the filtering error dynamics. Our approach can be extended to discrete systems and uncertain systems.

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作者简介:

杨富文 (1963 —),男,1990 年于华中理工大学自动控制系获博士学位,现为福州大学电气工程系教授,博士生导师,主要研究方向是 H_∞控制与滤波,信号处理,迭代学习控制以及自动化工程应用研究,E-mail: fwyang@fzu.edu.cn.