Article ID: 1000-8152(2003)05-0741-05

Robust H_∞ filtering of stochastic uncertain systems

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Abstract: The robust H_∞ estimation under parametric and stochastic uncertainties was studied. It was assumed that the uncertain parameter was norm bounded, the exogenous disturbance was stochastic uncertain and the systems were expressed by Itô's stochastic differential equations. The H_∞ filtering was constructed via solving a linear matrix inequality, and an example was presented to illustrate the developed theory.

Key words: stochastic H_∞ filtering; uncertainty; Wiener process; linear matrix inequality
CLC number: TP13 **Document code**: A

随机不确定系统的鲁棒 H_∞ 滤波

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摘要: 研究在同时具有参数和随机不确定的情况下的鲁棒 H_∞ 估计问题.假设系统的方程由 Itô 随机微分方程描述,不确定的参数是范数有界的,外部干扰是随机不确定的.通过解一个线性矩阵不等式,可以设计鲁棒 H_∞ 滤波器,最后给出的一个例子对理论分析进行了阐述.

关键词: 随机 H_∞ 滤波; 不确定性; Wiener 过程; 线性矩阵不等式

1 Introduction

This paper deals with the robust H_∞ state estimation problem of stochastic uncertain systems. Stochastic H_∞ control and design (including discrete and continuous systems) have recently received a great deal of attention (see [1 ~ 4], etc, and at the references therein). In [2], a stochastic bounded real lemma was developed, based on which, the output feedback H_∞ estimation has been derived for continuous-time stochastic uncertain systems, the solution depends on two coupled nonlinear matrix inequalities, the model in [2] does not include uncertainty in measurement equation, and the state equation does not include uncertain parameters in system matrix. In [4], the robust H_∞ control was discussed interms of complete state measurement.

Recently, [1] deals with a very general robust H_∞ filtering estimation problem of stationary continuous-time linear systems with stochastic uncertainties appearing both in state and measurement equations. A necessary and sufficient condition for H_∞ filtering was obtained via linear matrix inequalities (LMIs), and at the same time, for mixed H_2/H_∞ filtering, a suboptimal solution was presented. This kind of model is often applied in filtered estimation.

The main contribution of this paper is as follows: First, the model we employ is more general than that of [1]. Here we futher allow the state matrix to be uncertain with norm bounded. Because in practice, an exact dynamical model is not easily obtained, this kind of assumption has been made by many authors in deterministic models (see [5,6] and the reference therein). Second, a robust H_∞ filtering is designed by solving an LMI. In contrast to [1], we can design not only a full-order but also a reduced-order observer. Third, compared with the results in terms of algebraic Riccati equation[4,7], our approach is more applicable to practical computation[8].

Received date: 2002-04-30; Revised date: 2002-11-20.
Foundation item: supported by the Natural Science Foundation of Shandong Province (Q99G01).

2 Problem setting

For the sake of convenience, we adopt the following notations:

A' represents the transpose of a matrix or vector A;

$A \geqslant 0 (A > 0)$ means A a positive semidefinite (positive definite) matrix;

$L^2(\mathbb{R}_+, \mathbb{R}^l)$: the space of nonanticipative stochastic processes $y(t) \in \mathbb{R}^l$ with respect to filter F_t satisfying $\| y(t) \|_{L_2}^2 := \mathrm{E}\int_0^\infty |y(t)|^2 \mathrm{d}t < \infty$;

I: identity matrix.

In addition, we make the following assumption:

Assumption 1 All matrices appeared in this paper are real constant.

Consider the following linear uncertain system

$$\begin{cases} \mathrm{d}x = ((A + \Delta A)x + B_1 w)\mathrm{d}t + Dx\mathrm{d}\beta, \\ \mathrm{d}y = (Cx + D_{21}w)\mathrm{d}t + Fx\mathrm{d}\xi, \\ z = Lx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $x(0) := x_0$ is any norm bounded vector, $w \in L^2(\mathbb{R}_+, \mathbb{R}^q)$ is the exogenous stochastic disturbance signal, $y(t) \in \mathbb{R}^r$ is the measurement output, $z \in \mathbb{R}^m$ is the state combination to be estimated. ΔA is any uncertain matrix with norm bounded satisfying

$$\Delta A = D_1 F_1(t) G_1,\ F_1'(t) F_1(t) \leqslant I,\ \forall t \geqslant 0.$$

$\beta(t), \xi(t)$ are mutually uncorrelated, normal scalar Wiener processes, defined on the probability space (Ω, F, P) relative to an increasing family $(F_t)_{t \in \mathbb{R}_+}$.

Taking the filter equation for the estimation of $z(t)$ as

$$\mathrm{d}\hat{x} = A_f \hat{x}\mathrm{d}t + B_f \mathrm{d}y,\ \hat{x}_0 = 0,\ \hat{z} = C_f \hat{x}, \tag{2}$$

where $\hat{x} \in \mathbb{R}^{n_f}, n_f \leqslant n, \hat{z} \in \mathbb{R}^m$. Let

$$\xi' = [x' \quad \hat{x}'],\ \tilde{z} = z - \hat{z}. \tag{3}$$

Based on the above representation, the filter estimation error can be seen as the output of the augmented system

$$\mathrm{d}\xi = \tilde{A}\xi\mathrm{d}t + \tilde{D}_1 \xi\mathrm{d}\beta + \tilde{D}_2 \xi\mathrm{d}\zeta + \tilde{B}w\mathrm{d}t,\ \tilde{z} = \tilde{C}\xi \tag{4}$$

where

$$\tilde{A} = \begin{bmatrix} A+\Delta A & 0 \\ B_f C & A_f \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix},\ \tilde{D}_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

$$\tilde{D}_2 = \begin{bmatrix} 0 & 0 \\ B_f F & 0 \end{bmatrix},\ \tilde{C} = [L, \quad -C_f]. \tag{6}$$

Remark 1 [1] demanded $n_f = n$, i.e., (2) is a full-order observer. Here, we drop this restriction.

For any given disturbance attenuation level $\gamma > 0$, define the H_∞ performance index as

$$J_s = \| \tilde{z}(t) \|_{L_2}^2 - \gamma^2 \| w(t) \|_{L_2}^2. \tag{7}$$

We give the following definitions:

Definition 1 We say that system

$$\mathrm{d}x(t) = (A + \Delta A)x\mathrm{d}t + Dx\mathrm{d}\beta \tag{8}$$

is robustly exponentially two-stable, if for any admissible uncertain matrix $F(t)$, there exist some positive constants ρ, e

$$\mathrm{E}\,|x(t)|^2 \leqslant \rho\,|x(0)|^2 \exp(-et).$$

It is well known[9] that (8) is robustly exponentially two-stable for $t \geqslant 0$ if there exists $V(t,x) \in C_2^0(\{t > 0\} \times \mathbb{R}^n)$ such that

$$k_1 |x|^2 \leqslant V(t,x) \leqslant k_2 |x|^2,$$

$$\mathscr{L}V(t,x) \leqslant -k_3 |x|^2$$

for some positive constants k_1, k_2, k_3, where $\mathscr{L}$ is so-called the infinitesinal generator of (8).

Definition 2 Stochastic uncertain system

$$\mathrm{d}x = ((A + \Delta A)x + B_1 w)\mathrm{d}t + Dx\mathrm{d}\beta,\ x_0 \in \mathbb{R}^n \tag{9}$$

is said to be robustly internally stable, if (9) with $w = 0$ is robustly exponentially two-stable.

As in [1], stochastic H_∞ filtering estimation problem can be formulated as follows: Given $\gamma > 0$, find an asymptotically stable linear filter of the form (2) leading to a mean square stable estimation error process $\tilde{z}$ such that $J_s < 0$ for all nonzero $w \in \mathscr{L}^2(\mathbb{R}_+, \mathbb{R}^q)$ with $x_0 = 0$.

3 Robust H_∞ filtering

In this section we deal with robust H_∞ estimation problem. Note that our system is actually time-varying, so stochastic bounded real lemma derived by [2] can not be used to cope with this case. First give two lemmas as follows:

Lemma 1[10] For any real matrices of appropriate dimensions X and Y, we have

$$X'Y + Y'X \leqslant X'X + Y'Y.$$

Lemma 2 (Schur's complement[8]) For real matrices $N, M = M', R = R' > 0$, the following two conditions are equivalent:

1) $$M + NR^{-1}N' < 0;$$

2) $$\begin{bmatrix} M & N \\ N' & -R \end{bmatrix} < 0.$$

Our main result is as follows

Theorem 1 For any given disturbance attenuation $\gamma > 0$ and the filter of (2), if the following matrix inequality

$$P\tilde{A}_{11} + \tilde{A}'_{11}P + \tilde{D}'_1 P\tilde{D}_1 + \tilde{D}'_2 P\tilde{D}_2 + \tilde{C}'\tilde{C} + \frac{1}{\gamma^2}P\tilde{B}\tilde{B}'P + P\tilde{D}_{11}\tilde{D}'_{11}P + \tilde{G}'_{11}\tilde{G}_{11} < 0 \quad (10)$$

has a solution $P > 0$, then (4) is internally stable and the H_∞ filtering performance $J_s < 0$, where

$$\tilde{A}_{11} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix},\ \tilde{D}_{11} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix},\ \tilde{G}_{11} = [G_1 \quad 0]. \quad (11)$$

Proof We first prove (4) is internally stable. Take the Lyapunov function $V(\xi) = \xi' P \xi$ with $P > 0$ a solution to (10), let $\mathscr{L}_0$ be the infinitesimal operator of (4) with $w = 0$, then

$$\begin{aligned} &\mathscr{L}_0 V(\xi) = \\ &\xi'(P\tilde{A} + \tilde{A}'P + \tilde{D}'_1 P\tilde{D}_1 + \tilde{D}'_2 P\tilde{D}_2)\xi = \\ &\xi'(P\tilde{A}_{11} + \tilde{A}'_{11}P + P\Delta A_{11} + \Delta A'_{11}P + \\ &\tilde{D}'_1 P\tilde{D}_1 + \tilde{D}'_2 P\tilde{D}_2)\xi, \end{aligned} \quad (12)$$

where

$$\Delta A_{11} = \begin{bmatrix} D_1 F_1(t) G_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

By means of Lemma 1, we have

$$\begin{aligned} &P\Delta A_{11} + \Delta A'_{11}P = \\ &P\tilde{D}_{11}F_1(t)\tilde{G}_{11} + \tilde{G}'_{11}F'_1(t)\tilde{D}'_{11}P \leqslant \\ &P\tilde{D}_{11}\tilde{D}'_{11}P + \tilde{G}'_{11}G_{11}, \end{aligned}$$

so

$$\begin{aligned} &\mathscr{L}_0 V(\xi) \leqslant \\ &\xi'(P\tilde{A}_{11} + \tilde{A}'_{11}P + \tilde{D}'_1 P\tilde{D}_1 + \\ &\tilde{D}'_2 P\tilde{D}_2 + P\tilde{D}_{11}\tilde{D}'_{11}P + \tilde{G}'_{11}\tilde{G}_{11})\xi. \end{aligned}$$

Obviously, if (10) holds, then there exists $k_3 > 0$, such that $\mathscr{L}_0 V(\xi) \leqslant -k_3 |\xi|^2$, which yields that (4) is robustly internally stable.

Second, we prove $J_s < 0$ for all nonzero $w \in \mathscr{L}^2(\mathbb{R}_+, \mathbb{R}^q)$ with $\xi(0) = 0$. Let $\mathscr{L}_1$ be the infinitesimal operator of (4), note that

$$\begin{aligned} &J_s(T) := \mathrm{E}\int_0^T (\tilde{z}'z - \gamma^2 w'w)\mathrm{d}t = \\ &\mathrm{E}\int_0^T [(\xi'\tilde{C}'\tilde{C}\xi - \gamma^2 w'w)\mathrm{d}t + d(\xi'P\xi) - d(\xi'P\xi)] = \\ &-\mathrm{E}\,\xi'(T)P\xi(T) + \\ &\mathrm{E}\int_0^T [(\xi'\tilde{C}'\tilde{C}\xi - \gamma^2 w'w) + \mathscr{L}_1 V(\xi)]\mathrm{d}t \leqslant \\ &\mathrm{E}\int_0^T [(\xi'\tilde{C}'\tilde{C}\xi - \gamma^2 w'w) + \mathscr{L}_0 V(\xi) + \\ &w'\tilde{B}'P\xi + \xi'P\tilde{B}w]\mathrm{d}t \leqslant \\ &\mathrm{E}\int_0^T [\xi'(P\tilde{A}_{11} + \tilde{A}'_{11}P + \tilde{D}'_1 P\tilde{D}_1 + \tilde{D}'_2 P\tilde{D}_2 + \\ &P\tilde{D}_{11}\tilde{D}'_{11}P + \tilde{G}'_{11}\tilde{G}_{11} + \tilde{C}'\tilde{C})\xi - \gamma^2 w'w]\mathrm{d}t + \\ &\mathrm{E}\int_0^T (w'\tilde{B}'P\xi + \xi'P\tilde{B}w)\mathrm{d}t = \\ &\mathrm{E}\int_0^T \begin{bmatrix} \xi \\ w \end{bmatrix}' \begin{bmatrix} \{P\tilde{A}_{11} + \tilde{A}'_{11}P + \tilde{D}'_1 P\tilde{D}_1 + \tilde{D}'_2 P\tilde{D}_2 + P\tilde{D}_{11}\tilde{D}'_{11}P + \tilde{G}'_{11}\tilde{G}_{11} + \tilde{C}'\tilde{C}\} & P\tilde{B} \\ \tilde{B}'P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \mathrm{d}t. \end{aligned}$$

Therefore, if

$$\begin{bmatrix} \{P\tilde{A}_{11} + \tilde{A}'_{11}P + \tilde{D}'_1 P\tilde{D}_1 + \tilde{D}'_2 P\tilde{D}_2 + P\tilde{D}_{11}\tilde{D}'_{11}P + \tilde{G}'_{11}\tilde{G}_{11} + \tilde{C}'\tilde{C}\} & P\tilde{B} \\ \tilde{B}'P & -\gamma^2 I \end{bmatrix} < 0, \quad (13)$$

then there exist $\varepsilon > 0$, $J_s(T) \leqslant -\varepsilon^2 \mathrm{E}\int_0^T |w|^2 \mathrm{d}t < 0$, which yields $J_s \leqslant -\varepsilon^2 \mathrm{E}\int_0^\infty |w|^2 \mathrm{d}t < 0$. By Lemma 2, (13) is equivalent to (10). This ends the proof.

Remark 2 (10) is a nonlinear matrix inequality, so it is difficult to solve. Next, we will show that, in some special case, it can be transformed into an LMI, which is easily computed by the existing software[11].

Theorem 2 If the follwing linear matrix inequalities

$$\begin{bmatrix} P_{11}A + A'P_{11} + G_1'G_1 & C'Z_1' & D'P_{11} & F'Z_1' & P_{11}B_1 & P_{11}D_1 & L' \\ Z_1C & Z + Z' & 0 & 0 & Z_1D_{21} & 0 & -C_f' \\ P_{11}D & 0 & -P_{11} & 0 & 0 & 0 & 0 \\ Z_1F & 0 & 0 & -P_{22} & 0 & 0 & 0 \\ B_1'P_{11} & D_{21}'Z_1' & 0 & 0 & -\gamma^2 I & 0 & 0 \\ D_1'P_{11} & 0 & 0 & 0 & 0 & -I & 0 \\ L & -C_f & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{14}$$

have solutions $P_{11} > 0, P_{22} > 0, C_f, Z_1, Z$, then (4) is internally stable and $J_s < 0$, while

$$\mathrm{d}\hat{x} = P_{22}^{-1}Z\hat{x}\,\mathrm{d}t + P_{22}^{-1}Z_1\mathrm{d}y,\ \hat{Z} = C_f\hat{x} \tag{15}$$

is the corresponding H_∞ filter.

Proof By Lemma 2, (10) is equivalent to

$$\begin{bmatrix} P\tilde{A}_{11} + \tilde{A}_{11}'P + \tilde{G}_{11}'\tilde{G}_{11} & \tilde{D}_1'P & \tilde{D}_2'P & P\tilde{B} & P\tilde{D}_{11} & \tilde{C}' \\ P\tilde{D}_1 & -P & 0 & 0 & 0 & 0 \\ P\tilde{D}_2 & 0 & -P & 0 & 0 & 0 \\ \tilde{B}'P & 0 & 0 & -\gamma^2 I & 0 & 0 \\ \tilde{D}_{11}'P & 0 & 0 & 0 & -I & 0 \\ \tilde{C} & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0. \tag{16}$$

Take $P = \mathrm{diag}\,(P_{11}, P_{22})$, substituting (5), (6), (11) into (16), we have

$$\begin{bmatrix} P_{11}A + A'P_{11} + G_1'G_1 & C'B_f'P_{22} & D'P_{11} & 0 & 0 & F'B_f'P_{22} & P_{11}B_1 & P_{11}D_1 & L' \\ P_{22}B_fC & P_{22}A_f + A_f'P_{22} & 0 & 0 & 0 & 0 & P_{22}B_fD_{21} & 0 & -C_f' \\ P_{11}D & 0 & -P_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_{11} & 0 & 0 & 0 & 0 \\ P_{22}B_fF & 0 & 0 & 0 & 0 & -P_{22} & 0 & 0 & 0 \\ B_1'P_{11} & D_{21}'B_f'P_{22} & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 & 0 \\ D_1'P_{11} & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ L & -C_f & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \tag{17}$$

which is equivalent to

$$\begin{bmatrix} P_{11}A + A'P_{11} + G'G_1 & G_1'B_f'P_{22} & D'P_{11} & F'B_f'P_{22} & P_{11}B_1 & P_{11}D_1 & L' \\ P_{22}B_fC & P_{22}A_f + A_f'P_{22} & 0 & 0 & P_{22}B_fD_{21} & 0 & -C_f' \\ P_{11}D & 0 & -P_{11} & 0 & 0 & 0 & 0 \\ P_{22}B_fF & 0 & 0 & -P_{22} & 0 & 0 & 0 \\ B_1'P_{11} & D_{21}'B_f'P_{22} & 0 & 0 & -\gamma^2 I & 0 & 0 \\ D_1'P_{11} & 0 & 0 & 0 & 0 & -I & 0 \\ L & -C_f & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0. \tag{18}$$

Let $P_{22}A_f = Z, P_{22}B_f = Z_1$, then (18) becomes (14). (15) is easily seen from our assumption and Theorem 2 is proved.

The following example illustrates how to design the reduced-order observer (2).

Example 1 In (1), we take

$$A = \begin{bmatrix} -7.5 & 0 \\ 0 & -14.2 \end{bmatrix}, \ B_1 = \begin{bmatrix} 2.1 \\ 1.8 \end{bmatrix},$$

$$D = \begin{bmatrix} -1.6 & 1.4 \\ 0.1 & -2.5 \end{bmatrix}, \ D_1 = \begin{bmatrix} 1.3 \\ 2.1 \end{bmatrix},$$

$G_1 = [1.4 \quad 1]$, $L = [-1 \quad 1]$, $C = [1.0 \quad -1]$, $D_{21} = 1.3$, $F = [1.5 \quad 2.6]$, $|F_1(t)| \leqslant 1$, $\gamma = 0.8$, then by software packages such as LMI optimization toolbox in Matlab[11], we can easily solve (14) and obtain

$$P_{11} = 0.5869, \ P_{22} = 21.0725,$$

$$Z = -12.1315, \ Z_1 = 0.1214, \ C_f = 1.6673.$$

So

$$A_f = P_{22}^{-1} Z = -6.0658, \ B_f = P_{22}^{-1} Z_1 = 0.0607.$$

Accordingly, the reduced-order H_∞ filtering (2) can be constructed as

$$d\hat{x} = -6.0658\hat{x}dt + 0.0607dy, \ \hat{z} = 1.6673\hat{x}.$$

4 Conclusion

This paper has discussed robust H_∞ estimation problems of stochastic uncertain systems. By comparison with [1], we allow the system parameter to be uncertain with norm bounded. Especially, we show that robust H_∞ filtering can be acheived if linear matrix inequality (14) is solvable. Since the design of H_∞ filtering is often encountered in some fields, such as signal processing, our results not only have practical value but also have feasibility. When $\Delta A = 0$, our models degenerate into those of [1].

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