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Stabilization of unstable equilibria of chaotic systems and its applications to Chua's circuit

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Abstract: Based on the ergodicity of chaos and the state PI regulator approach, a new method was proposed for stabilizing unstable equilibria and for tracking set-point targets for a class of chaotic systems with nonlinearities satisfying a specific condition. A criterion was derived for designing the controller gains, in which control parameters could be selected by solving a Lyapunov matrix inequality. In particular, for piecewise linear chaotic systems, such as Chua's circuit, the control parameters can be selected via the pole placement technique in linear control theory. More importantly, this method has high robustness to system parametric variations and strong rejection to external constant-disturbances. For verification and demonstration, the design method is applied to the chaotic Chua's circuit, showing satisfactory simulation results.

Key words: Chua's circuit; unstable equilibrium point; stabilization; PI regulator

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混沌系统不稳定平衡点的镇定及其在蔡氏电路中的应用

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摘要: 基于混沌系统的遍历性和状态 PI 调节器理论,提出一类混沌系统不稳定平衡点的镇定和设定点跟踪新方法,给出用于控制器参数设计的 Lyapunov 矩阵不等式.对于分段线性混沌系统,如蔡氏电路,可通过控制理论中的极点配置技术来设计控制器参数.该方法对系统参数变化具有很强的鲁棒性,能够消除外部定值扰动.将该方法用于蔡氏混沌电路不稳定平衡点的镇定,取得了满意的结果.

关键词: 蔡氏电路; 不稳定平衡点; 镇定; PI 调节器

1 Introduction

In the past decade, much attention has been paid to chaos control, and many methods have been proposed for suppressing chaos[1,2]. For instance, the delayed feedback control (DFC) method[3] is based on the difference between the current system output and the time-delayed output signals, which does not require any knowledge of the target points. However, this approach in general cannot specify the target setting point and is subject to the so-called odd number eigenvalue limitation[4~6]. On the other hand, the OGY method[7], which is a local control scheme, and the methods[8,9] that are based on precise state feedback control usually fail with system parameters variation and are inconvenient for practical engineering systems.

In this paper, based on the ergodicity of chaos and state PI regulator approach[10], a feedback control design method is proposed for stabilizing unstable equilibria and for set-point tracking for a class of chaotic systems with nonlinearities satisfying a specific condition. The proposed method combines a state feedback and an integral of the difference between the target output and the current output signals. The output signal is a simple function (e.g., linear combination) of the state variables of the chaotic system. In particular, if a suitable linear combination is selected and used as the output feedback,

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the target output signal can become zero, and then no information about the target equilibrium is needed in the integral part of the controller. Moreover, this control method has satisfactory control performance and robustness. It will also be demonstrated that this control method can reject external bounded constant-disturbances asymptotically.

Based on the Lyapunov stabilization theory, a criterion is derived for choosing the proportional and integral gains. The control parameters can be selected via solving a Lyapunov matrix inequality. In particular, for piecewise linear chaotic systems, such as Chua's circuit, the control parameters can be chosen via the pole placement technique in linear control theory.

2 Stabilizing unstable equilibria of a class of chaotic systems

Consider a controlled chaotic system of the form

$$\dot{x} = Ax + g(x) + u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^n$ is the control input to be designed, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, and $g(x)$ is a continuous nonlinear function satisfying the following condition[11]:

$$g(x) - g(\tilde{x}) = M_{x,\tilde{x}}(x - \tilde{x}), \tag{2}$$

where $M_{x,\tilde{x}}$ is a bounded matrix that depends on both x and $\tilde{x}$.

Remark 1 Many chaotic systems can be described by (1) and (2), such as the classic Chua's circuit[12], the modified Chua's circuit with a sine function, the modified Chua's circuit with nonlinear quadratic function $x|x|$[13], and the MLC circuit.

Let x_s be an unstable equilibrium of (1) when $u = 0$, that is,

$$Ax_s + g(x_s) = 0. \tag{3}$$

The objective is to design a controller u such that the states of system (1) are stabilized to x_s, which is a constant vector independent of time. Later, the objective will also be extended to tracking a constant set-point.

According to the state PI regulator theory, a controller is constructed as follows:

$$u = \lambda[B(K(x - x_s) + k\int_0^t (y - y_s)\mathrm{d}\tau)], \tag{4}$$

where $B \in \mathbb{R}^{n\times 1}$ is a constant gain matrix, $K \in \mathbb{R}^{1\times n}$ is the proportional state feedback gain vector, $k \in \mathbb{R}$ is the integral gain, $y = Cx$ is the output with a constant matrix $C \in \mathbb{R}^{1\times n}$, $y_s = Cx_s$ is the observation of the target equilibrium x_s, and

$$\lambda = \begin{cases} 1, & \text{if } x \in \Omega_{x_s}, \\ 0, & \text{else}, \end{cases} \tag{5}$$

where Ω_{x_s} denotes the neighborhood of the unstable equilibrium x_s.

Remark 2 Because of the ergodicity of chaos, the trajectory will visit or access Ω_{x_s} at times. When the trajectory accesses Ω_{x_s}, the controller (4) is turned on, and the trajectory will converge to x_s asymptotically under the controller (4), in which the control parameters will be chosen to ensure the error dynamic system is asymptotically stable, as further described below.

Remark 3 If a matrix $C \in \mathbb{R}^{1\times n}$ is chosen appropriately such that $y_s = Cx_s = 0$, the controller structure (4) can be further simplified, as further discussed later.

Now, it follows from Eqs. (1) ~ (5) that the error dynamics system is obtained as

$$\begin{aligned}\dot{e} =\ & \\ & (A+\lambda BK)e+\lambda kB\int_0^t(y-y_s)\mathrm{d}\tau+g(x)-g(x_s)) = \\ & (A+\lambda BK)e+\lambda kB\int_0^t(y-y_s)\mathrm{d}\tau+M_{x,x_s}(x-x_s) = \\ & (A + \lambda BK + M_{x,x_s})e + \lambda kB\int_0^t(y - y_s)\mathrm{d}\tau,\end{aligned} \tag{6}$$

where $e = x - x_s$.

Let $q = \int_0^t (y - y_s)\mathrm{d}\tau$, then $\dot{q} = y - y_s = Ce$, so that (6) can be reformulated as the following incremental error state equation:

$$\begin{bmatrix}\dot{e}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}(A + M_{x,x_s} + \lambda BK) & \lambda Bk \\ C & 0\end{bmatrix}\begin{bmatrix}e\\ q\end{bmatrix}, \tag{7}$$

or

$$\dot{\bar{e}} = (\bar{A} + \lambda\bar{B}\cdot\bar{K})\bar{e}, \tag{8}$$

where $\bar{e} = \begin{bmatrix}e\\ q\end{bmatrix}$, $\bar{A} = \begin{bmatrix}A + M_{x,x_s} & 0\\ C & 0\end{bmatrix}$, $\bar{B} = \begin{bmatrix}B\\ 0\end{bmatrix}$, $\bar{K} = [K \quad k]$.

Theorem 1 If there exists a positive definite and symmetric constant matrix P such that

$$(\bar{A} + \lambda\bar{B}\cdot\bar{K})^{\mathrm{T}}P + P(\bar{A} + \lambda\bar{B}\cdot\bar{K}) \leqslant \mu I < 0 \tag{9}$$

uniformly for all x in the phase space, where μ denotes a negative constant and I is the identity matrix, then the zero solution of the error dynamics system (8) is globally exponentially stable. Consequently, the chaotic system (1) can be stabilized at the equilibrium x_s by controller (4).

Proof Choose the Lyapunov function

$$V = \bar{e}^{\mathrm{T}} P \bar{e}, \tag{10}$$

where P is a positive definite and symmetric constant matrix. Then, its derivative is

$$\begin{aligned}\dot{V} &= \dot{\bar{e}}^{\mathrm{T}} P \bar{e} + \bar{e}^{\mathrm{T}} P \dot{\bar{e}} = \\ &((\bar{A} + \lambda \bar{B} \cdot \bar{K})\bar{e})^{\mathrm{T}} P \bar{e} + \bar{e}^{\mathrm{T}} P (\bar{A} + \lambda \bar{B} \cdot \bar{K})\bar{e} = \\ &\bar{e}^{\mathrm{T}}((\bar{A} + \lambda \bar{B} \cdot \bar{K})^{\mathrm{T}} P + P(\bar{A} + \lambda \bar{B} \cdot \bar{K}))\bar{e} \leqslant \\ &\mu \| \bar{e} \|^2 < 0,\end{aligned} \tag{11}$$

where $\| \cdot \|$ denotes the Euclidean norm. Based on the Lyapunov stability theory, system (8) is globally exponentially stable.

Remark 4 In Theorem 1, condition (9) can be further simplified to be

$$(\bar{A} + \bar{B} \cdot \bar{K})^{\mathrm{T}} P + P(\bar{A} + \bar{B} \cdot \bar{K}) \leqslant \mu I < O,$$

where $x \in \Omega_{x_s}$ and $\bar{A} = \begin{bmatrix} A + M_{x,x_s} & 0 \\ C & 0 \end{bmatrix}$. If M_{x,x_s} is a constant matrix when $x \in \Omega_{x_s}$, denoted by M, then $\bar{A}$ is a constant matrix. Hence, the feedback gain matrix $\bar{K} = [K \quad k]$ can be selected by the pole placement technique. In this case, B and C are selected such that $\begin{bmatrix} A + M & B \\ C & 0 \end{bmatrix}$ is nonsingular and $(A + M, B)$ is controllable. As a result, $(\bar{A}, \bar{B})$ is controllable, and the eigenvalues of $(\bar{A} + \bar{B} \cdot \bar{K})$ can be arbitrarily placed by selecting appropriate values for K and k. Here, note that

$$[\bar{B} \quad \bar{A}\bar{B} \quad (\bar{A})^2 \bar{B} \quad \cdots \quad (\bar{A})^n \bar{B}] =$$

$$\begin{bmatrix} A + M & B \\ C & 0 \end{bmatrix} \begin{bmatrix} 0 & B & (A + M)B & \cdots & (A + M)^{n-1} B \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Remark 5 In control law (4), if we choose $y_s = r$, where r is a constant set-point for tracking, then the output y can track this set-point asymptotically.

Remark 6 If there exists an external bounded constant-disturbance w, whose value is unknown but bounded, in the system (1), then we can easily prove that the chaotic system can be stabilized at the targeted unstable equilibrium point by using the similar procedure above.

3 Application to Chua's circuit

To illustrate the controller design method outlined above and to show its advantages, the well-known Chua's circuit is used here as an example.

The chaotic Chua's circuit is described by[12]

$$\begin{cases} \dot{x} = \alpha(y - x - f(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y, \end{cases} \tag{12}$$

where $\alpha > 0, \beta > 0, a < -1 < b < 0, f(\cdot)$ is a piecewise linear function defined by

$$f(x) = bx + \frac{1}{2}(a - b)(| x + 1 | - | x - 1 |). \tag{13}$$

For the piecewise-linear function $f(\cdot)$ in (13), we have

$$f(x) - f(\tilde{x}) = k_{x,\tilde{x}}(x - \tilde{x}), \tag{14}$$

where $k_{x,\tilde{x}}$ is the slope of the linear segment, depending on both x and $\tilde{x}$, and varies within the interval $[a, b]$ for all $t \geqslant 0$, i.e., is bounded by $a \leqslant k_{x,\tilde{x}} \leqslant b < 0$.

Let $x_s = [x_s \quad y_s \quad z_s]^{\mathrm{T}}$ be an unstable equilibrium of (12), satisfying

$$\begin{cases} \alpha(y_s - x_s - f(x_s)) = 0, \\ x_s - y_s + z_s = 0, \\ -\beta y_s = 0. \end{cases} \tag{15}$$

By solving (15), three equilibria can be found:

$$P_1\left(\frac{a - b}{1 + b}, 0, -\frac{a - b}{1 + b}\right), \quad P_2(0,0,0),$$

and

$$P_3\left(-\frac{a - b}{1 + b}, 0, \frac{a - b}{1 + b}\right).$$

The chaotic Chua circuit, under the control of the state PI regulator, is described by

$$\dot{x} = Ax + g(x) + \lambda B\left(K(x - x_s) + k\int_0^t (Cx - Cx_s)\mathrm{d}\tau\right), \tag{16}$$

where $A = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, C = [c_1 \quad c_2 \quad c_3], K = [k_1 \quad k_2 \quad k_3], x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and

$g(x) = \begin{bmatrix} -\alpha f(x) \\ 0 \\ 0 \end{bmatrix}$. We have

$$g(x) - g(x_s) = \begin{bmatrix} -\alpha(f(x) - f(x_s)) \\ 0 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} -\alpha k_{x,x_s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = M_{x,x_s} e, \tag{17}$$

where $M_{x,x_s} = \begin{bmatrix} -\alpha k_{x,x_s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. According to Eq.(8), the corresponding error dynamics system state equation is

$$\dot{\bar{e}} = (\bar{A} + \lambda \bar{B} \cdot \bar{K})\bar{e}, \qquad (18)$$

where $\bar{e} = \begin{bmatrix} e \\ q \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A + M_{x,x_s} & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} -\alpha(1+k_{x,x_s}) & \alpha & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -\beta & 0 & 0 \\ c_1 & c_2 & c_3 & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix}$, $\bar{K} = [K \quad k] = [k_1 \quad k_2 \quad k_3 \quad k]$.

For $P_1\left(\frac{a-b}{1+b}, 0, -\frac{a-b}{1+b}\right)$, we have $x_s = \frac{a-b}{1+b} < -1$. Let $\Omega_{P_1} = \{x : x \leqslant -1\}$. If $x \in \Omega_{P_1}$, then

$$k_{x,x_s} = b, \quad M_{x,x_s} = \begin{bmatrix} -\alpha b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{A} = \begin{bmatrix} -\alpha(1+b) & \alpha & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -\beta & 0 & 0 \\ c_1 & c_2 & c_3 & 0 \end{bmatrix}.$$

According to Theorem 1 and Remark 6, we have:

Corollary 1 If B, C, K and k are selected such that $(\bar{A} + \bar{B} \cdot \bar{K})$ is Hurwitz stable for all $x \in \Omega_{P_1}$, then the zero solution of the error dynamics system (18) is globally stable, and consequently the chaotic system (12) can be stabilized at the first equilibrium x_s.

Using the same procedure, we can also obtain the similar results for P_2 and P_3.

4 Simulations

In this section, the stabilization of the first equilibrium, $P_1\left(\frac{a-b}{1+b}, 0, -\frac{a-b}{1+b}\right)$, is demonstrated. The parameters of Chua's circuit (12) used are $\alpha = 9.78$, $\beta = 14.97$, $a = -1.31$, $b = -0.75$, so the system exhibits chaotic behavior[12].

Based on the chosen equilibrium and Corollary 1, after calculation, we selected

$$B = [0 \quad 0 \quad 1]^{T}, \quad C = [0 \quad 1 \quad 0],$$
$$K = [-1.4045 \quad 0.9755 \quad -3.8550]$$

and $k = -4.4761$,

so that the eigenvalues of matrix $(\bar{A} + \bar{B} \cdot \bar{K})$ are -1.6, -1.9, -2.0 and -1.8. The controlled Chua's system is stabilized at the intended equilibrium, as shown in Figs. 1, 2 and 3 for different initial conditions. In Fig. 1, the initial value is $x_0(-1.6, -1.2, -1.5)$, which belongs to Ω_{P_1} while the initial value $x_0(0.1, 0.12, 0.5)$ in Fig. 2 belongs to Ω_{P_2} instead; in Fig. 3, the initial value $x_0(1.6, 1.2, 1.5)$ belongs to Ω_{P_3}.

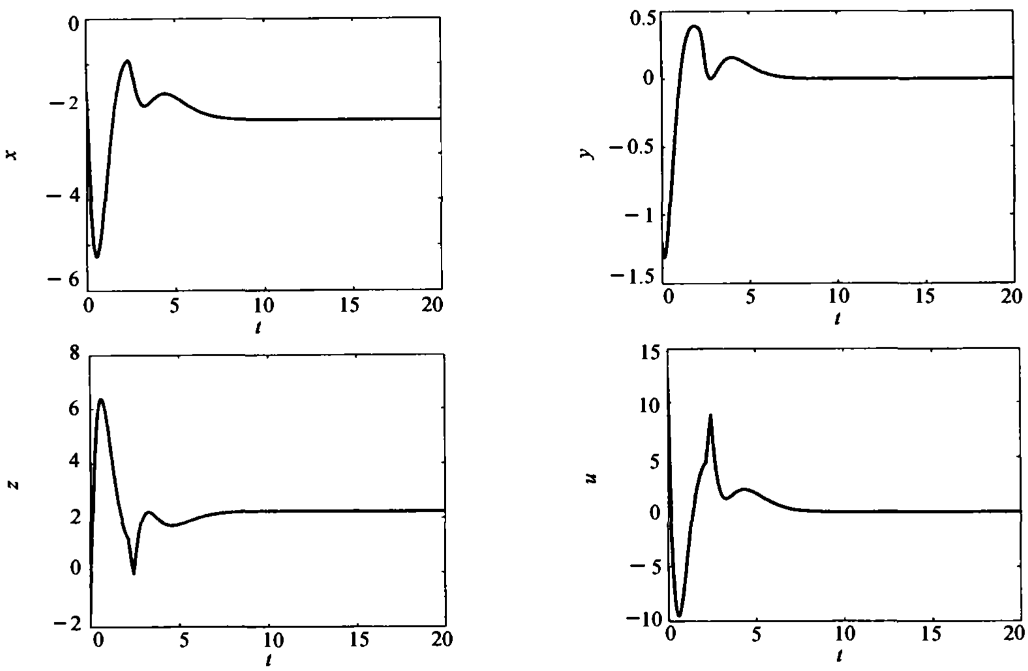

Fig. 1 Stabilization of Chua's circuit with $x_0 \in \Omega_{P_1}$

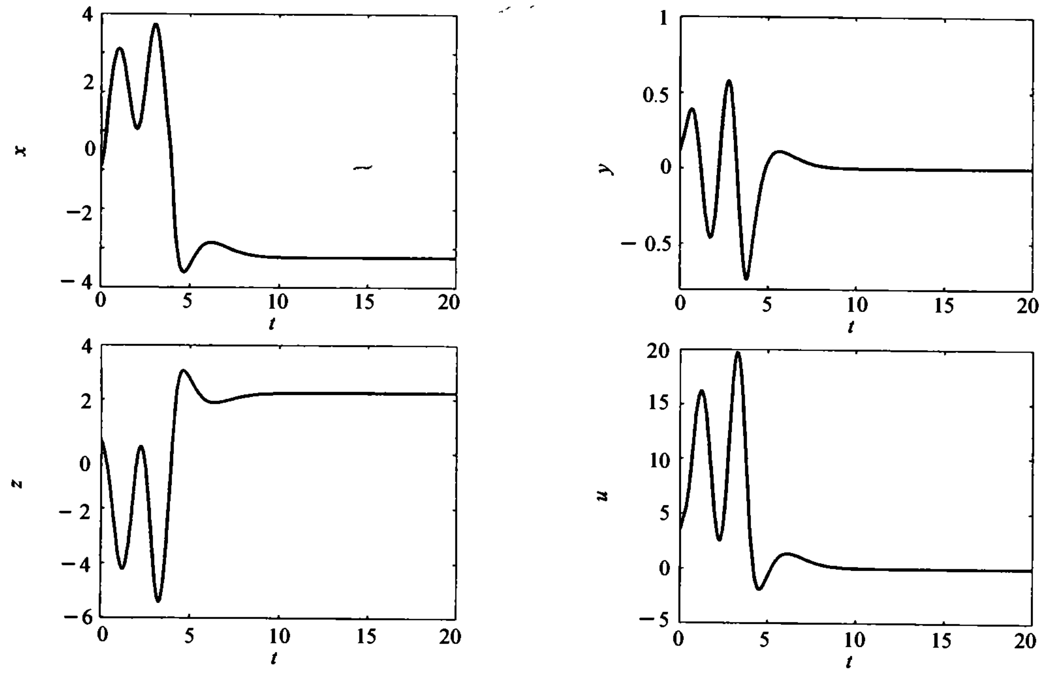

Fig. 2 Stabilization of Chua's circuit with $x_0 \in \Omega_{P_2}$

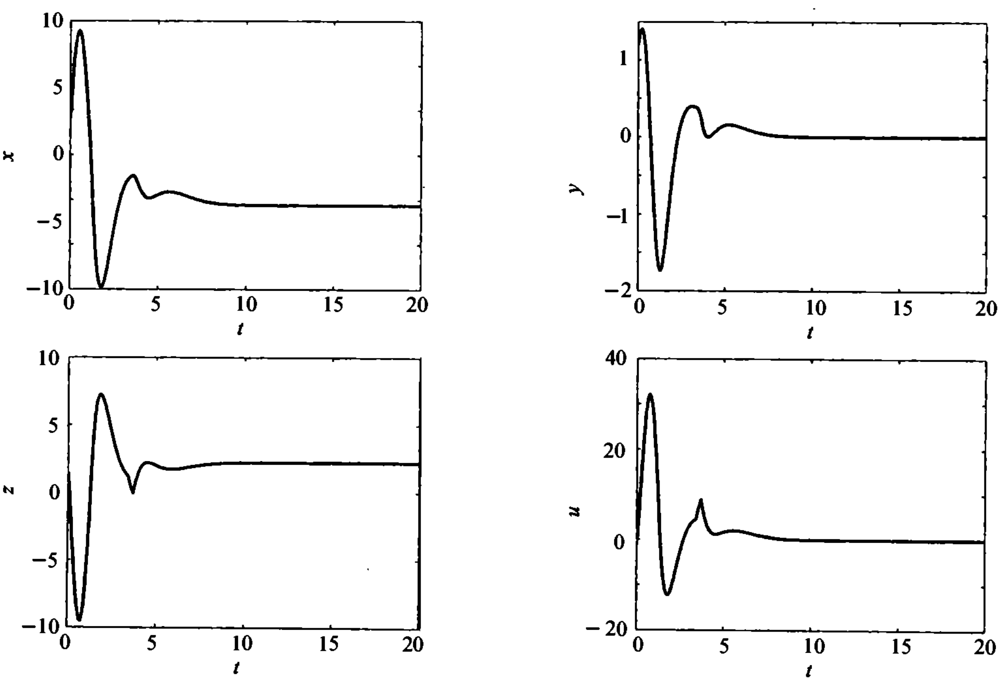

Fig. 3 Stabilization of Chua's circuit with $x_0 \in \Omega_{P_3}$

To demonstrate the robustness of the control performance, let there be an external bounded disturbance, w, in the third equation of Chua's circuit, with $w = 0.2$. The system, although with the disturbance, can still be stabilized at the targeted equilibrium by the same controller, as shown in Fig. 4. To verify the robustness to parameters variation in system (12), the system parameters are changed to, say, $\alpha = 9.0, \beta = 14.87, a = -1.27, b = -0.68$. It is demonstrated in Fig.5 that the modified system can still be stabilized at the first equilibrium by the same controller, as expected.

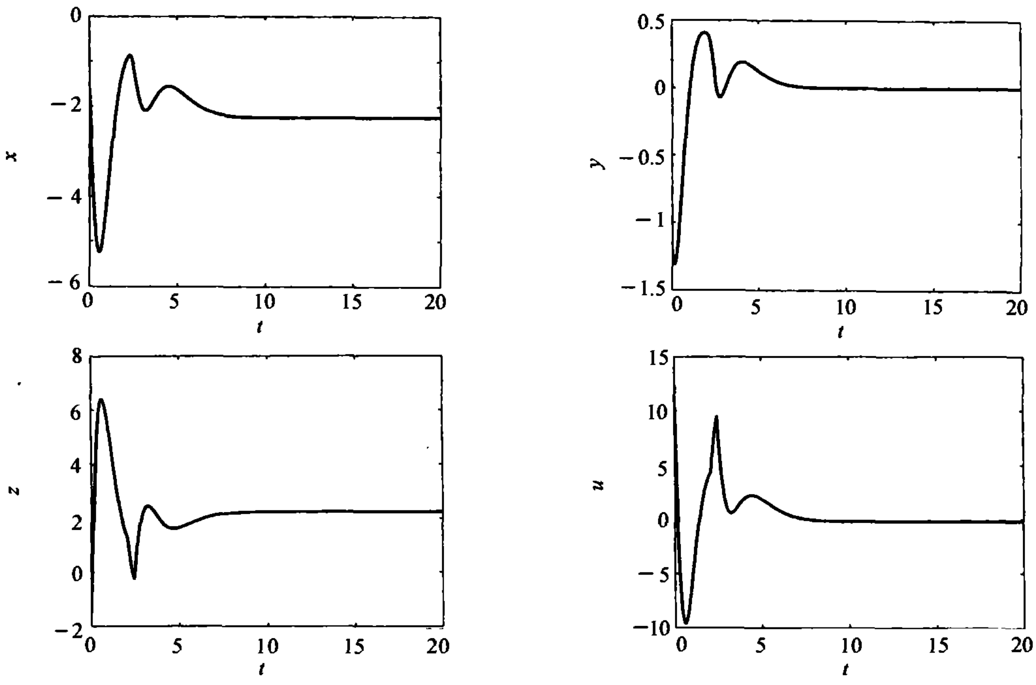

Fig. 4 Stabilization of Chua's circuit with $w = 0.2$

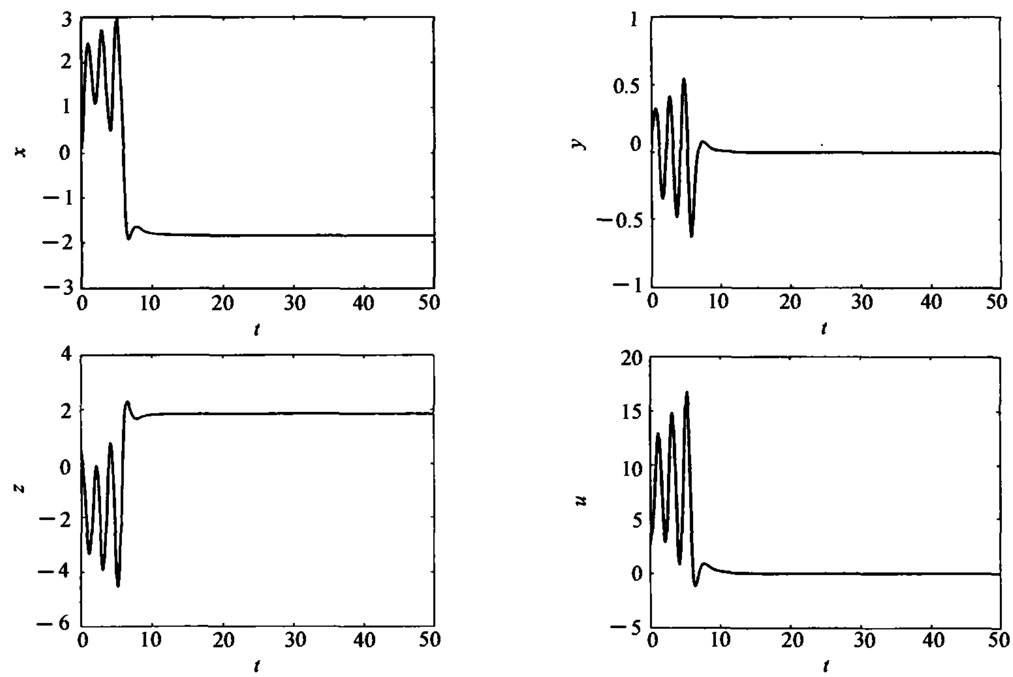

Fig. 5 Stabilization of Chua's circuit with different parameters

5 Conclusion and discussion

In this paper, a new method for stabilizing unstable equilibria has been developed for a class of chaotic systems based on the state PI regulator method. The proposed method is robust to a certain level of external disturbances as well as system parameters variation. Based on the Lyapunov stabilization theory, a precise criterion is derived to accomplish the stabilization of the target unstable equilibria of the chaotic system. The control parameters can be selected via solving a Lyapunov matrix inequality. Particularly, for piecewise linear chaotic systems such as Chua's circuit, they can be selected via the simple pole placement technique. This new design method is better than the state feedback control method in the sense that even the given chaotic system has significant parameters variation, the designed controller remains to be effective for stabilizing its orbit to the desired target equilibrium. This method is also better than Pyragas' delay feedback control method since it is guaranteed to control to a specified target point.

In principle, the method and criterion proposed here can be applied to various chaotic systems with nonlinear functions satisfying condition (2) given in this paper. Therefore, similar designs can be carried out for the chaotic systems such as the modified Chua's circuit with a sine function, the modified Chua's circuit with the quadratic function $x|x|$, the MLC circuit, etc.

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