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基于Lyapunov能量函数的倒立摆稳定控制

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摘要：倒立摆是一种复杂的非线性控制系统.通过对其进行控制能够检验控制器的鲁棒性.基于一个能量形式的Lyapunov函数设计了倒立摆稳定控制器使得摆趋于上平衡位置,并且使得小车位移和角度都收敛于零.该控制策略基于系统的总能量,利用其耗散特性设计了Lyapunov函数,并证明了控制系统的稳定性.理论分析及仿真试验表明该控制器对于倒立摆控制具有很强的鲁棒性.

关键词：非线性控制；鲁棒控制；能量；倒立摆；鲁棒性

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Lyapunov-energy based stability control for inverted pendulums

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Abstract: The inverted pendulum is a complicated nonlinear control system. The robustness of a control can be verified by controlling it effectively. A stability controller was first designed for inverted pendulums based on a Lyapunov function with the form of energy, the pendulum was then raised to its upper equilibrium position and the cart displacement and angle were both brought to zero. The control scheme was based upon the total energy of the system, and the dissipative characteristic was used to design a Lyapunov function, the stability of the control system was also proved. Through the analysis and simulation, it was manifested that the controller has strong robustness.

Key words: nonlinear control; robust control; energy; inverted pendulum, robustness

1 引言(Introduction)

在现代控制理论中,倒立摆是一个普遍用于教学的试验仪器.摆以小车上一固定点自由摆动,控制目标是通过在水平面上移动小车,使得摆最终位于向上垂直方向的非稳定平衡位置.一般的非线性控制方法不能控制倒立摆系统.Jakubczyk和Respondek[1]证明了倒立摆系统不能进行反馈线性化.Lin[2]等人提出了一种线性控制器来稳定具有约束轨迹的倒立摆的线性化模型,然而用这种线性控制器用于倒立摆的非线性模型控制时,需要确定吸引域.Mazenc和Praly[3]提出了一个具有附加积分项的倒立摆控制器,与其他方法不同的是,他们的方法是最终使得小车的位移向零收敛.作者针对文献[4]中所提出的基准问题,设计了一个基于能量形式的Lyapunov函数控制器,该控制器对于倒立摆的稳定控制具有很强的鲁棒性.

2 系统动态模型(System dynamic model)

摆沿水平方向的运动方程为

$$f_H = m\frac{d^2}{dt^2}[x(t) + L\sin\varphi(t)]. \tag{1}$$

台车水平方向的运动方程为

$$M\frac{d^2x(t)}{dt^2} = f(t) - f_H(t) - \delta(\dot{x}). \tag{2}$$

其中 $\delta(\dot{x})$ 表示摩擦力,且有 $|\delta(\dot{x})| \leqslant \gamma|\dot{x}|$. 另外,

$$f_F(t) = f_H(t). \tag{3}$$

其中 $f_F(t)$ 为水平方向上的推力.所以小台车运动方程为

$$\begin{aligned} M\frac{d^2x(t)}{dt^2} &= \\ f(t) - m\frac{d^2}{dt^2}[x(t) + L\sin\varphi(t)] - \delta(\dot{x}) &= \\ f(t) - m\frac{d^2x(t)}{dt^2} - (mL\cos\varphi(t))\frac{d^2\varphi(t)}{dt^2} &= \\ (mL\sin\varphi(t))\left(\frac{d\varphi(t)}{dt}\right)^2 - \delta(\dot{x}). \end{aligned} \tag{4}$$

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即

$$(M+m)\ddot{x}(t)+(mL\cos\varphi(t))\ddot{\varphi}(t)-(mL\sin\varphi(t))(\dot{\varphi}(t))^2=f(t)-\delta(\dot{x}) \tag{5}$$

倒立摆系统的动态可由如下方程描述:

$$(J+mL^2)\ddot{\varphi}(t)+(mL\cos\varphi(t))\ddot{x}(t)+(C_0+\Delta C)\dot{\varphi}(t)-mLg\sin\varphi(t)=0. \tag{6}$$

其中 $|\Delta C|\leqslant\bar{C}$. 式(5)与式(6)合并,可以写成

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=\tau. \tag{7}$$

其中

$$q=\begin{bmatrix}x\\ \varphi\end{bmatrix},$$

$$M(q)=\begin{bmatrix}M+m & mL\cos\varphi(t)\\ mL\cos\varphi(t) & J+mL^2\end{bmatrix}, \tag{8}$$

$$C(q,\dot{q})=\begin{bmatrix}0 & -mL\dot{\varphi}(t)\sin\varphi(t)\\ 0 & C_0+\Delta C\end{bmatrix}, \tag{9}$$

$$G(q)=\begin{bmatrix}0\\ -mLg\sin\varphi(t)\end{bmatrix},\ \tau=\begin{bmatrix}f-\delta(\dot{x})\\ 0\end{bmatrix}. \tag{10}$$

注意到 $M(q)$ 为对称阵,并且

$$\det(M(q))=(M+m)(J+mL^2)-(mL\cos\varphi(t))^2=M(J+mL^2)+mJ+m^2L^2-m^2L^2\cos^2\varphi(t)=M(J+mL^2)+mJ+m^2L^2\sin^2\varphi(t)>0. \tag{11}$$

小车摆的总能量为

$$E=K(q,\dot{q})+P(q)=\frac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q}+mgL(\cos\varphi-1). \tag{12}$$

由式(7)~(10),可得

$$\begin{aligned}\dot{E}&=\dot{q}^{\mathrm{T}}M(q)\ddot{q}+\frac{1}{2}\dot{q}^{\mathrm{T}}\dot{M}(q)\dot{q}-mgL\dot{\varphi}\sin\varphi=\\ &\dot{q}^{\mathrm{T}}M(q)\ddot{q}+\frac{1}{2}\dot{q}^{\mathrm{T}}\dot{M}(q)\dot{q}+\dot{q}^{\mathrm{T}}G(q)=\\ &\dot{q}^{\mathrm{T}}\Big(M(q)\ddot{q}+\frac{1}{2}\dot{M}(q)\dot{q}\Big)+\dot{q}^{\mathrm{T}}G(q)=\\ &\dot{q}^{\mathrm{T}}\Big(-C(q,\dot{q})\dot{q}+\tau+\frac{1}{2}\dot{M}(q)\dot{q}\Big)=\\ &\dot{q}^{\mathrm{T}}\tau-(C_0+\Delta C)\dot{\varphi}^2=\\ &\dot{x}(f-\delta(\dot{x}))-(C_0+\Delta C)\dot{\varphi}^2.\end{aligned} \tag{13}$$

3 鲁棒控制器的设计(Design of robust controller)

设 Lyapunov 函数为

$$V(q,\dot{q})=\frac{k_E}{2}E^2+\frac{1}{2}k_{x1}x^2+\frac{1}{2}k_{x2}\dot{x}^2. \tag{14}$$

其中 k_E,k_{x1},k_{x2} 均为大于零的常数.则

$$\begin{aligned}\dot{V}(q,\dot{q})&=k_EE\dot{E}+k_{x1}x\dot{x}+k_{x2}\dot{x}\ddot{x}=\\ &\dot{x}(k_EE(f-\delta(\dot{x}))+k_{x2}\ddot{x}+k_{x1}x)-\\ &k_EE(C_0+\Delta C)\dot{\varphi}^2.\end{aligned} \tag{15}$$

由式(8)和式(11)可得

$$M^{-1}(q)=\frac{1}{\det(M(q))}\begin{bmatrix}J+mL^2 & -mL\cos\varphi(t)\\ -mL\cos\varphi(t) & M+m\end{bmatrix}. \tag{16}$$

其中

$$\det(M(q))=(M+m)J+mL^2(M+m\sin^2\varphi(t)).$$

因此,可以得到

$$\begin{aligned}\begin{bmatrix}\ddot{x}\\ \ddot{\varphi}\end{bmatrix}&=[\det(M(q))]^{-1}\times\\ &\Bigg(-\begin{bmatrix}J+mL^2 & -mL\cos\varphi(t)\\ -mL\cos\varphi(t) & M+m\end{bmatrix}\times\\ &\begin{bmatrix}0 & -mL\dot{\varphi}(t)\sin\varphi(t)\\ 0 & C_0+\Delta C\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\varphi}\end{bmatrix}-\\ &\begin{bmatrix}J+mL^2 & -mL\cos\varphi(t)\\ -mL\cos\varphi(t) & M+m\end{bmatrix}\begin{bmatrix}0\\ -mLg\sin\varphi(t)\end{bmatrix}+\\ &\begin{bmatrix}J+mL^2 & -mL\cos\varphi(t)\\ -mL\cos\varphi(t) & M+m\end{bmatrix}\begin{bmatrix}f-\delta(\dot{x})\\ 0\end{bmatrix}\Bigg)=\\ &[\det(M(q))]^{-1}\times\\ &\left\{\begin{bmatrix}0 & (J+mL^2)mL\dot{\varphi}\sin\varphi+(C_0+\Delta C)mL\cos\varphi\\ 0 & -m^2L^2\dot{\varphi}\sin\varphi\cos\varphi-(M+m)(C_0+\Delta C)\end{bmatrix}\times\right.\\ &\begin{bmatrix}\dot{x}\\ \dot{\varphi}\end{bmatrix}+\begin{bmatrix}-m^2L^2g\sin\varphi\cos\varphi\\ (M+m)mLg\sin\varphi\end{bmatrix}+\\ &\left.\begin{bmatrix}(J+mL^2)(f-\delta(\dot{x}))\\ -mLf\sin\varphi+mL\cos\varphi\delta(\dot{x})\end{bmatrix}\right\}.\end{aligned} \tag{17}$$

并且,由文献[4]中给出的转动惯量 $J=\dfrac{mL^2}{3}$,可得

$$\begin{aligned}\ddot{x}&=\frac{1}{(M+m)J+mL^2(M+m\sin^2\varphi(t))}\times\\ &[(J+mL^2)mL\dot{\varphi}^2\sin\varphi+\\ &(C_0+\Delta C)mL\dot{\varphi}\cos\varphi-m^2L^2g\sin\varphi\cos\varphi+\\ &(J+mL^2)(f-\delta(\dot{x}))]=\\ &\frac{1}{(M+m)/3+(M+m\sin^2\varphi(t))}\times\\ &[(4mL\dot{\varphi}^2\sin\varphi)/3+((C_0+\Delta C)\dot{\varphi}\cos\varphi)/L-\\ &mg\sin\varphi\cos\varphi+4(f-\delta(\dot{x}))/3].\end{aligned} \tag{18}$$

所以

$$\begin{aligned}&\dot{V}(q,\dot{q})=\\ &\dot{x}(k_EE(f-\delta(\dot{x}))+\\ &k_{x2}\frac{1}{(M+m)/3+(M+m\sin^2\varphi(t))}\times\\ &[(4mL\dot{\varphi}\sin\varphi)/3+((C_0+\Delta C)\dot{\varphi}\cos\varphi)/L-\\ &mg\sin\varphi\cos\varphi+4(f-\delta(\dot{x}))/3]+k_{x1}x)-\end{aligned}$$

$$k_E E(C_0+\Delta C)\dot{\varphi}^2=$$
$$\dot{x}\left(\frac{k_E E((4M+m)/3+m\sin^2\varphi)+4k_{x2}/3}{(M+m)/3+(M+m\sin^2\varphi)}\times\right.$$
$$(f-\delta(\dot{x}))+\frac{(4mL\dot{\varphi}^2\sin\varphi)/3+((C_0+\Delta C)\dot{\varphi}\cos\varphi)/L}{(M+m)/3+(M+m\sin^2\varphi(t))}-$$
$$\left.\frac{mg\sin\varphi\cos\varphi}{(M+m)/3+(M+m\sin^2\varphi(t))}+k_{x1}x\right)-$$
$$-k_E E(C_0+\Delta C)\dot{\varphi}^2=$$
$$\dot{x}\left(\frac{k_E E((4M+m)/3+m\sin^2\varphi)+4k_{x2}/3}{(M+m)/3+(M+m\sin^2\varphi)}f-\right.$$
$$\frac{k_E E((4M+m)/3+m\sin^2\varphi)+4k_{x2}/3}{(M+m)/3+(M+m\sin^2\varphi)}\delta(\dot{x})+$$
$$\frac{(4mL\dot{\varphi}^2\sin\varphi)/3+(C_0\dot{\varphi}\cos\varphi)/L}{(M+m)/3+(M+m\sin^2\varphi(t))}-$$
$$\frac{mg\sin\varphi\cos\varphi}{(M+m)/3+(M+m\sin^2\varphi(t))}+$$
$$\left.\frac{(\Delta C\dot{\varphi}\cos\varphi)/L}{(M+m)/3+(M+m\sin^2\varphi(t))}+k_{x1}x\right)-$$
$$k_E E(C_0+\Delta C)\dot{\varphi}^2. \tag{19}$$

要使得上述 Lyapunov 函数的导数为负值,只要选取

$$f=\frac{(M+m)/3+(M+m\sin^2\varphi)}{k_E E((4M+m)/3+m\sin^2\varphi)+4k_{x2}/3}\times$$
$$\left\{-\frac{k_E E((4M+m)/3+m\sin^2\varphi)+4k_{x2}/3}{(M+m)/3+(M+m\sin^2\varphi)}\gamma\dot{x}-\right.$$
$$\frac{(4mL\dot{\varphi}^2\sin\varphi)/3+(C_0\dot{\varphi}\cos\varphi)/L}{(M+m)/3+(M+m\sin^2\varphi(t))}+$$
$$\frac{mg\sin\varphi\cos\varphi}{(M+m)/3+(M+m\sin^2\varphi(t))}-$$
$$\left.\frac{\bar{C}|\dot{\varphi}\cos\varphi|/L}{(M+m)/3+(M+m\sin^2\varphi(t))}\mathrm{sgn}(\dot{x})-k_{x1}x\right\}. \tag{20}$$

其中 f 为驱动力.

由于

$$f=\frac{K_T K_g}{R_a\gamma}v(t)-\frac{K_T^2K_g^2}{R_a r^2}\dot{x}, \tag{21}$$

所以,可得

$$v(t)=\frac{R_a r}{K_T K_g}\left(f+\frac{K_T^2K_g^2}{R_a r^2}\dot{x}\right). \tag{22}$$

由上面的分析及式(7)和式(10)可以看出,在整个倒立摆系统力矩控制项中,只要设计对小车的驱动力满足 Lyapunov 稳定性,即可对整个倒立摆系统进行控制.

4 仿真(Simulation)

对于模型(7)选取参数为

$M=0.455, m=0.210, K_T=K_E=7.767\times10^{-3}$, $R_a=2.6, r=6.35\times10^{-3}, K_g=3.7, g=9.8$, $C_0=0.06, L=0.305, \bar{C}=0.06, \gamma=0.95$, $k_E=4, k_{x1}=k_{x2}=1, \varphi(0)=\pi/4, x(0)=1$.

于是有

$$M(q)=\begin{bmatrix}0.455+0.210 & 0.210\times0.305\cos\varphi(t)\\ 0.210\times0.305\cos\varphi(t) & 4\times0.210\times0.305^2/3\end{bmatrix},$$

$$C(q,\dot{q})=\begin{bmatrix}0 & -0.210\times0.305\dot{\varphi}(t)\sin\varphi(t)\\ 0 & 0.06+\Delta C\end{bmatrix},$$

$$G(q)=\begin{bmatrix}0\\ -0.210\times0.305\times9.8\sin\varphi(t)\end{bmatrix},$$

$$E=\frac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q}+0.210\times9.8\times0.303(\cos\varphi-1),$$

$$f=$$
$$\frac{(0.455+0.210)/3+(0.455+0.210\sin^2\varphi)}{4E((4\times0.455+0.210)/3+0.210\sin^2\varphi)+4/3}\times$$
$$\left\{-\frac{4E((4\times0.455+0.210)/3+0.210\sin^2\varphi)+4/3}{(0.455+0.210)/3+(0.455+0.210\sin^2\varphi)}\times\right.$$
$$0.950\dot{x}-$$
$$\frac{(4\times0.210\times0.305\dot{\varphi}^2\sin\varphi)/3}{(0.455+0.210)/3+(0.455+0.210\sin^2\varphi(t))}-$$
$$\frac{(0.06\dot{\varphi}\cos\varphi)/0.305}{(0.455+0.210)/3+(0.455+0.210\sin^2\varphi(t))}+$$
$$\frac{0.210\times9.8\sin\varphi\cos\varphi}{(0.455+0.210)/3+(0.455+0.210\sin^2\varphi(t))}-$$
$$\frac{0.06|\dot{\varphi}\cos\varphi|/0.305}{(0.455+0.210)/3+(0.455+0.210\sin^2\varphi(t))}\times$$
$$\left.\mathrm{sgn}(\dot{x})-x\right\}$$

所以,

$$v(t)=\frac{R_a r}{K_T K_g}\left(f+\frac{K_T^2K_g^2}{R_a r^2}\dot{x}\right).$$

小车位移与摆的角度曲线如图 1~3 所示.

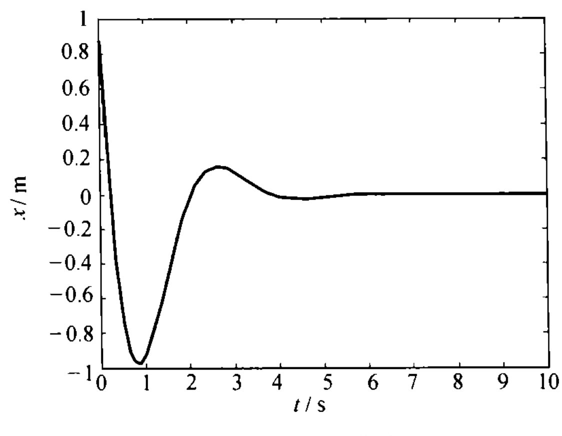

图 1　小车位移曲线

Fig. 1　Position moving curve of cart

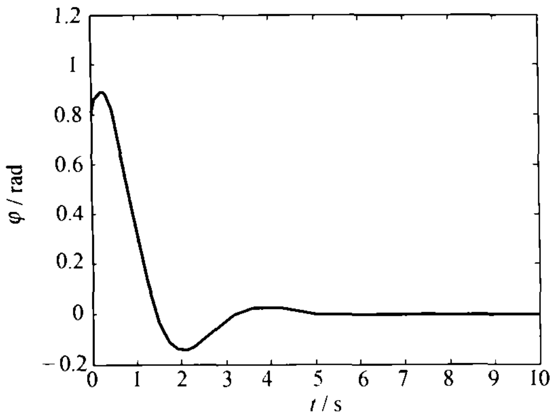

图 2 摆的角度曲线

Fig. 2 Angle curve of pendulum

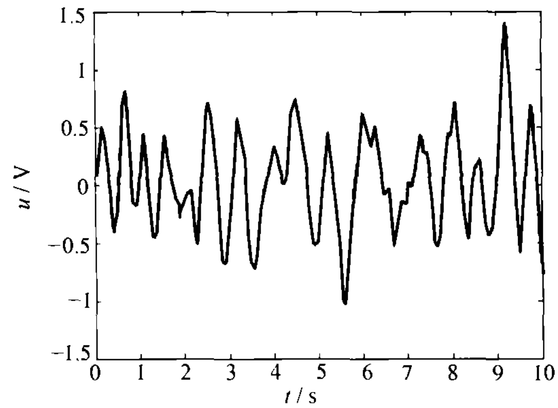

图 3 控制输入曲线

Fig. 3 Control input curve

5 结论(Conclusion)

本文中所设计的控制器能使得小车与摆角都趋于零,控制策略基于系统的总能量,并且利用其耗散特性设计了 Lyapunov 函数.理论分析及仿真可以看出,这种控制方法对于倒立摆系统具有较快的收敛速度和很强的鲁棒性.

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