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增长率为输出的未知多项式非线性系统的全局输出反馈跟踪

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摘要: 本文研究了一类增长线性地依赖于不可测状态非线性系统的输出反馈自适应实用跟踪问题. 很不同的是, 本文所研究系统的增长率是输出的未知多项式(系数未知、幂次已知), 且关于被跟踪参考信号的假设相当弱(仅本身和其导数为已知的), 为解决该问题, 通过灵活采用通用控制和死区的思想和方法, 引入了带有新型动态增益的观测器来重构不可测的系统状态, 进而构造了自适应输出反馈跟踪控制器. 可以证明, 当控制器中的设计参数适当选取时, 闭环系统所有状态有界, 并且跟踪误差趋于事先给定的充分小的区域. 数值仿真说明了所提方法的有效性.

关键词: 非线性系统; 增长依赖于不可测状态; 输出未知多项式的增长率; 全局实用跟踪; 动态高增益; 输出反馈

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Global output-feedback tracking for nonlinear systems with unknown polynomial-of-output growth rate

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Abstract: This paper is concerned with the adaptive practical tracking by output-feedback for a class of nonlinear systems with linearly unmeasurable states dependent growth. Quite different, the system growth rate is unknown polynomial-of-output with given powers but unknown coefficients, and the assumption on the to-be-tracked reference signal is rather weak (merely itself and its time-derivative are known). To solve the problem, by flexibly using the ideas and methods from universal control and dead zone, an observer with new dynamic high-gain is first introduced to re-construct the unmeasurable system states, and then an adaptive output-feedback tracking controller is successfully designed. It is shown that if the design parameters in controller are suitably chosen, then all the states of the closed-loop system are bounded, and furthermore, the tracking error will be prescribed sufficiently small when time is large enough. A numerical simulation is also provided to demonstrate the validity of the proposed approaches.

Key words: nonlinear systems; unmeasurable states dependent growth; unknown polynomial-of-output growth rate; global practical tracking; dynamic high-gain; output-feedback

1 Introduction and problem formulation

Global output tracking is a research issue of theoretical and practical importance, and has received much attention during last two decades for many classes of nonlinear systems, by incorporating with adaptive technique, backstepping design, dead zone and output regulation theory, among others (see e.g., [1–17], as well as the references therein). With sufficient condition/information on nonlinear systems and the reference signal to be tracked, *asymptotic output tracking* can be achieved, see, e.g., [4, 8–9, 11]. However, when condition/information is not sufficient, this type of control would be very hard to achieve, even impossible, and consequently, practical output tracking, which is enough for many practical applications, is proposed to establish a slightly more degraded control objective than the former one; that is, the tracking error is steered prescribed small when time is large enough, rather than asymptotically convergent to zero. Besides, practical output tracking usually needs less information than asymptotic output tracking, and particularly, allows the presence of many classes of unmodeled dynamics and uncertainties/unknowns in the systems and the reference signal. Mainly because of these, practical output tracking have acquired much attention and is still an active area of research.

Recently, when only partial system states or output available for feedback, some representative results have been obtained for practical output tracking for classes of nonlinear systems in [7, 12–13, 15, 17], not only extending the related results on stabilization, but also developing distinct methodologies of control design and performance analysis. More specifically, in [13], practical output tracking was considered for a class of stochastic nonlinear sys-

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tems. Work[7] addressed adaptive practical output tracking (or λ-tracking) of nonlinear systems with unknown control coefficient and the growth of polynomial-of-output multiplying an unknown constant, and developed the backstepping design procedure. Different from [7], work[12] further studied the systems with the dominating nonlinearities linearly composed by unmeasurable states with a factor of bounded function, and presented a much simpler controller than that in [7]. Work[15], with less information on the reference signal than that in [7, 12], addressed the practical output tracking for nonlinear systems with higher-order unmeasurable states dependent growth. Our work[17] allows serious unknowns (i.e., unknown constant growth rate and unknown bounds for reference signal and its derivative), and hence essentially different from work[15]. It is worth mentioning that both in [7, 12, 17], a dead zone is employed in the updating law in feedback design to effectively restrain the bursting phenomenon, and more importantly, to directly establish the desired tracking objective.

This paper continues our investigation presented in [17], from unknown constant growth rate to unknown polynomial-of-output one. Specifically, the adaptive practical tracking is investigated for the following class of single-input single-output (SISO) nonlinear systems*:

$$\begin{cases} \dot{\zeta}_i = \zeta_{i+1} + \phi_i(t, \zeta, u),\ i = 1, \cdots, n-1, \\ \dot{\zeta}_n = u + \phi_n(t, \zeta, u), \\ y = \zeta_1 - y_{\mathrm{r}}, \end{cases} \tag{1}$$

where $n \in \mathbb{N}\backslash\{1\}$ is the system order; $\zeta = [\zeta_1\ \ \cdots\ \ \zeta_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the system state with the initial condition $\zeta(t_0) = \zeta_0$; $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the control input and system output, respectively; $y_{\mathrm{r}} : [t_0, +\infty) \to \mathbb{R}$ is the reference signal to be tracked by the system output; functions $\phi_i : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, i = 1, \cdots, n$ are continuous in the first argument and locally Lipschitz in the rest two arguments. In what follows, suppose only the system output y is measurable. This means that ζ_1 and y_{r} may not be measurable and hence the problem to be solved is different from those in [4, 9, 13] where y_{r} is precisely known. In addition, such situation is often encountered in practical applications, as already discussed in [15]. For notational simplicity, let $t_0 = 0$ in the later development of the paper.

The purpose of this paper is to search for an output-feedback controller such that the global practical tracking problem of system (1) can be solved under the following assumptions:

A1) There exist a known integer $p \in \mathbb{N}$ and an unknown constant $\theta \geqslant 0$, such that

$$|\phi_i(t, \zeta, u)| \leqslant \theta(1 + |\zeta_1|^p) \sum_{j=1}^{i} |\zeta_i| + \theta,$$

for $i = 1, \cdots, n$.

A2) The reference signal $t \mapsto y_{\mathrm{r}}(t)$, $t \in \mathbb{R}^+$ is continuously differentiable. Moreover, there exists an unknown constant $\vartheta \geqslant 0$ such that

$$\sup_{t \geqslant 0}(|y_{\mathrm{r}}(t)| + |\dot{y}_{\mathrm{r}}(t)|) \leqslant \vartheta.$$

Rigorously speaking, in order to establish the tracking objective, we will explicitly construct an adaptive output-feedback controller for system (1) under assumptions A1)) and A2) in the following from:

$$\dot{\chi} = \alpha_\lambda(\chi, y),\ u = \beta_\lambda(\chi, y), \tag{2}$$

such that

i) the solution of the resulting closed-loop system is well-defined and globally bounded on $[0, +\infty)$;

ii) for any prescribed constant $\lambda > 0$ and for any initial condition ζ_0, there is a finite time $T_\lambda > 0$ such that $\sup\limits_{t \geqslant T_\lambda} |y(t)| = \sup\limits_{t \geqslant T_\lambda} |\zeta_1(t) - y_{\mathrm{r}}(t)| \leqslant \lambda$,

where χ is the state vector with the appropriate dimension and the initial value $\chi_0 = \chi(0)$, $\lambda > 0$ is used to represent the tracking accuracy, functions α_λ and β_λ are vector-valued continuous and scalar continuously differentiable, respectively, both dependent on λ. The control just formulated is sometimes called *λ-tracking* (see e.g., [1–3, 12, 14] and the references therein).

It is worthwhile to point out assumption A1) shows that system (1) heavily relies on the unmeasurable states and has the growth rate of polynomial-of-output multiplying an unknown constant (or saying unknown coefficients polynomial-of-output). This assumption also makes system (1) is essentially different from those studied in [12, 18–20], where the authors considered classes of nonlinear systems with another type of unmeasurable states dependent growth, i.e., the system nonlinearities are consisted of unmeasurable states multiplying an unknown constant or functions satisfying some severe conditions. Besides, assumption A1) clearly means that the system may not necessarily have equilibrium points and allows the presence of the lower-order (than 1) growing unmeasurable states. From these, one can see that system (1) under assumption A1) represents a larger class of nonlinear systems and is significantly different from those in the existing literature on tracking control (see e.g., [4, 7, 9–13, 15, 17]). Assumption A2) shows that no more information is needed on the reference signal y_{r} except the existence (unnecessarily known) of the upper bounds of it and its derivative. Although with the similar but slightly stronger constraints than assumption A1), the stabilization problem has been settled in [21–22], the practical tracking problem has remained unsolved so far for system (1) under assumptions A1) and A2). This is partially because of the weaker conditions imposed on the system and reference signal. In addition, to author's knowledge, it seems impossible to realize the global asymptotical tracking control (see e.g., [7, 12, 15–16, 21, 23–24]).

*Throughout this paper, $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of all natural numbers; $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ denotes the set of all non-negative numbers, $\mathbb{R}^n$ denotes the real n-dimensional space; for any vector or matrix X, X^{T} denotes its transpose, and $\|X\|$ (i.e., $\|X\|_2$) and $\|X\|_\infty$ denote the Euclidean norm (or 2-norm) and the infinity norm (or maximum norm) for vectors, and the corresponding induced norm for matrices, respectively; for any symmetric matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote its maximum and minimum eigenvalues, respectively.

The paper proposes a new adaptive output-feedback controller which successfully accomplishes the global practical tracking objective prescribed above for system (1). First, an appropriate high-gain observer is constructed, and the high-gain contains two dynamic (updated on-line) components to compensate for the polynomial-of-function and unknown constant in the system growth rate, respectively (roughly speaking). Then, stimulated by the ideas and methods from universal control and dead zone[10, 12, 21–23], an adaptive output-feedback tracking controller is successfully constructed. Most importantly, the proof of the validness of the controller is rigorously addressed, which reveals that by a suitable choice of the design parameters, the controller designed can guarantee global practical output tracking, while keeping the global boundedness of all the closed-loop system states.

The remainder of the paper is organized as follows. Section 2 provides the design scheme for global practical tracking control, and Section 3 summarizes the main results of the paper. Section 4 gives a numerical example, and Section 5 addresses some concluding remarks. The paper ends with an important appendix which collects rigorous proofs of a crucial lemma and two fundamental propositions and obviously is an absolutely necessary part of the paper.

2 Output-feedback tracking control design

This section is devoted to designing an adaptive output-feedback tracking controller for system (1). First, a dynamic high-gain observer is introduced to reconstruct the system unmeasurable states. The novel updating laws for the high gains, which are inspired by the idea of dead zone and the related stabilization results, can effectively compensate the unknowns in the system and reference signal. Then, based on the dynamic high-gain observer, an adaptive output-feedback controller is explicitly designed to make the tracking error prescribed small after a finite time, while keeping the global boundedness of all the resulting closed-loop system states.

From $\zeta_1 = y + y_{\rm r}$, and assumptions A1) and A2), we know that, for any $i = 1, \cdots, n$, and any $t \in \mathbb{R}^+$, $\zeta \in \mathbb{R}^n$, $u \in \mathbb{R}$,

$$\begin{aligned}&|\phi_i(t,\zeta,u)| \leqslant \\ &\theta(1+|y+y_{\rm r}|^p)\sum_{j=1}^{i}|\zeta_i| + \theta \leqslant \\ &\theta(1+2^{p-1}\vartheta^p + 2^{p-1}|y|^p)\sum_{j=1}^{i}|\zeta_i| + \theta \leqslant \\ &\theta\max\{1+2^{p-1}\vartheta^p,\ 2^{p-1}\}(1+|y|^p)\sum_{j=1}^{i}|\zeta_i| + \theta.\end{aligned} \tag{3}$$

This obviously means that system (1) does not necessarily have an equilibrium point since the presence of adding θ in (3), and system (1) can be dominated by a system having linearly unmeasurable states dependent growth with the rate of polynomial-of-output multiplying an unknown constant.

2.1 State transformation

For the convenience of control design, let's first introduce the following simple transformation:

$$x_1 = y = \zeta_1 - y_{\rm r},\ x_i = \zeta_i,\ i = 2, \cdots, n. \tag{4}$$

Then, by (1), we have

$$\begin{cases}\dot{x}_i = x_{i+1} + \varphi_i(t,x,u),\ i = 1,\cdots,n-1,\\ \dot{x}_n = u + \varphi_n(t,x,u),\\ y = x_1,\end{cases} \tag{5}$$

where $x = [x_1\ \cdots\ x_n]^{\rm T}$ with the specified initial condition, and

$$\begin{cases}\varphi_1(t,x,u) = \phi_1(t,\zeta,u)\big|_{(4)} - \dot{y}_{\rm r},\\ \varphi_i(t,x,u) = \phi_i(t,\zeta,u)\big|_{(4)},\ i = 2,\cdots,n,\end{cases}$$

and for the simplicity of expression, variable t in φ_i's is used to denote all the effects caused by time t itself, the reference signal and its first derivative. Clearly, by assumption A2) and the definitions of φ_i's, we know that φ_i's satisfy the similar relations to (3) with respect to argument x, and therefore, the transformed system (5) has the similar growth property to that of the original system (1).

It should be emphasized that, by introducing transformation (4), the tracking control of the original system (1) will be solved by studying the following control design problem of the transformed system (5): an output-feedback controller should be designed such that all the closed-loop system states are bounded on $[0, +\infty)$ and meanwhile the system output is regulated into a prescribed small neighborhood of the origin when time is large enough.

2.2 Output-feedback control design

This subsection is to design an output-feedback controller in the form of (2) for system (5).

First, motivated by [21–22], the following state observer is constructed for system (5):

$$\begin{cases}\dot{\hat{x}}_i = \hat{x}_{i+1} + r^i(t)l_i(x_1 - \hat{x}_1),\\ \qquad i = 1,\cdots,n-1,\\ \dot{\hat{x}}_n = u + r^n(t)l_n(x_1 - \hat{x}_1),\end{cases} \tag{6}$$

where l_i's are constants to be determined later, $r(t) \triangleq L(t)M(t)$, $t \in \mathbb{R}^+$ is called dynamic gain in which L and M satisfy the following updating laws:

$$\begin{cases}\dot{L} = M\max\{0, \dfrac{2(y-\hat{x}_1)^2}{r^{2a}(t)} - \dfrac{\lambda^2}{2r(t)} + \\ \qquad 2\displaystyle\sum_{i=1}^{n}\dfrac{\hat{x}_i^2}{r^{2i-2+2a}(t)}\},\ L(0) = 1,\\ \dot{M} = -\beta_1 M^2 + \beta_2(1+|y|^p)^2 M,\\ \qquad M(0) = 1,\end{cases} \tag{7}$$

with to-be-determined design parameters β_1, β_2 and a satisfying $0 < \beta_1 \leqslant \beta_2$ and $0 < a < \dfrac{1}{10p}$, respectively (p is the same positive integer as in the assumption A2).

Thus, based on observer (6) and updating laws (7), the output-feedback controller is designed as follows:

$$u = -(r^n(t)k_1\hat{x}_1 + r^{n-1}(t)k_2\hat{x}_2 + \cdots + r(t)k_n\hat{x}_n), \tag{8}$$

where k_i's are constants to be determined later.

As to be stated in the later Proposition 1 and Lemma 1, one can see that $L(t) \geqslant 1,\ \forall t \in \mathbb{R}^+$ and $M(t) \geqslant 1,\ \forall t \in \mathbb{R}^+$, and consequently $r(t) \geqslant 1,\ \forall t \in \mathbb{R}^+$, and therefore, observer (6) is of the Luenberger-like one with the dynamic high-gain r. This type of observers are more flexible than the observers with the non-high-gain or constant-gain, and are particularly applicable to output-feedback control design for the systems with inherent nonlinearities and uncertainties [12, 15, 21, 24].

Roughly speaking, the components L and M of dynamic gain r play different important roles in realizing global practical tracking by output-feedback. On one hand, L will be updated large enough to compensate the bounded unknowns/uncertainties in system (5) (or system (1)) and the reference signal. On the other hand, the dynamic gain M is necessary to compensate the polynomial-of-output system growth rate.

It is worth pointing out that the particular updating law for L is different from those in the closely related literature (see e.g., [12, 21]). This is quite important, since the presence of dead zone in the updating law of L, ones can finally assert the prescribed tracking objective as long as the global stability is ensured for the closed-loop system, as to be stated in the proof of Theorem 1. Besides, from (8), one can know that the proposed controller is linearly composed of the observer states and hence easy to be implemented in practice. However, the subsequent treatment indicates that the stability and tracking performance analysis for the resulting closed-loop system is rather complicated (majorally due to the particular updating laws for L and M).

For the further treatment, define the state estimation error $\tilde{x} = x - \hat{x}$ of the resulting closed-loop system, which obviously satisfies

$$\begin{cases} \dot{\tilde{x}}_i = -r^i(t) l_i \tilde{x}_1 + \tilde{x}_{i+1} + \varphi_i, \\ \qquad i = 1, \cdots, n-1, \\ \dot{\tilde{x}}_n = -r^n(t) l_n \tilde{x}_1 + \varphi_n. \end{cases}$$

Besides, for the sake of simplicity, we introduce the following scaling transformations:

$$\begin{cases} \varepsilon_i = \dfrac{\tilde{x}_i}{r^{a+i-1}},\ i = 1, \cdots, n, \\ \eta_i = \dfrac{\hat{x}_i}{r^{a+i-1}},\ i = 1, \cdots, n, \end{cases} \tag{9}$$

where a is the same design parameter as in (7). Letting $\varepsilon = [\varepsilon_1\ \cdots\ \varepsilon_n]^{\mathrm{T}}$ and $\eta = [\eta_1\ \cdots\ \eta_n]^{\mathrm{T}}$, we have

$$\begin{cases} \dot{\varepsilon} = r(t) A \varepsilon + f - \dfrac{\dot{r}(t)}{r(t)} D_{\mathrm{a}} \varepsilon, \\ \dot{\eta} = r(t) B \eta + r(t) l \varepsilon_1 - \dfrac{\dot{r}(t)}{r(t)} D_{\mathrm{a}} \eta, \end{cases} \tag{10}$$

where

$$\begin{aligned} & l = [l_1\ \cdots\ l_n]^{\mathrm{T}}, \\ & f = [\frac{\varphi_1}{r^a}\ \frac{\varphi_2}{r^{a+1}}\ \cdots\ \frac{\varphi_n}{r^{a+n-1}}]^{\mathrm{T}} = \\ & \quad [\frac{\phi_1 - \dot{y}_{\mathrm{r}}}{r^a}\ \frac{\phi_2}{r^{a+1}}\ \cdots\ \frac{\phi_n}{r^{a+n-1}}]^{\mathrm{T}}, \end{aligned}$$

$$D_{\mathrm{a}} = \mathrm{diag}\{a,\ a+1,\ a+n-1\},$$

and

$$A = \begin{bmatrix} -l_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_{n-1} & 0 & \cdots & 1 \\ -l_n & 0 & \cdots & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix}.$$

3 Main results

In this section, the main contributions of the paper will be addressed and rigorously proven. As an additional consequence of the proofs, we will accomplish the entire conditions on the design parameters, and will deeply understand the necessity of the aforementioned choices/requirements, such as, the updating laws for L and M, and the preliminary constraints on $a,\ \beta_1,\ \beta_2$, i.e., $0 < a \leqslant \dfrac{1}{10p}$ and $0 < \beta_1 \leqslant \beta_2$.

Theorem 1 Consider system (1) under the assumptions A1) and A2). If the design parameters l_i's, k_i's, β_1, β_2 and a are suitably chosen, then based on the dynamic high-gain observer (6), the output-feedback controller (8) guarantees that all the states of the resulting closed-loop system are well-defined and bounded on $[0, +\infty)$, and furthermore, the global practical tracking can be achieved, i.e., for prescribed $\lambda > 0$, there exists a finite time T_λ, such that $|y(t)| \leqslant \lambda,\ \forall t \geqslant T_\lambda$.

Proof In fact, in the entire proof, the design parameters $l_i,\ k_i,\ i = 1, \cdots, n$ are chosen to satisfy (A.1), and a, β_1, β_2 are chosen to meet (A.6).

By (1), (5)–(8) and Statement iii) of Proposition 1 later, it is easy to verify that the right-hand side of the resulting closed-loop system is locally Lipschitz in $(x, \hat{x}, M, L)$ in an open neighborhood of the initial condition, and hence the closed-loop system has a unique solution on a small interval $[0, t_{\mathrm{f}})$ (see Theorem 3.1, page 18 of [25]). Let $[0,\ T_{\mathrm{f}})$ be the maximal interval on which a unique solution exists, where $0 < T_{\mathrm{f}} \leqslant +\infty$ (see Theorem 2.1, page 17 of [26]). As will be stated in Lemma 1 later where $T_{\mathrm{f}} = +\infty$, the closed-loop system states are well-defined on $[0, +\infty)$.

To continue the proof, we shall need some basic properties of components L and M in dynamic high-gain r, given in the following proposition.

Proposition 1 For system (5) and observer (6), the dynamic gains L and M described by (7) are provided with the following properties:

i) $M(t) \geqslant 1$ for any $t \in [0,\ T_{\mathrm{f}})$;

ii) L is monotone nondecreasing on $[0,\ T_{\mathrm{f}})$, and consequently $L(t) \geqslant 1$ for any $t \in [0,\ T_{\mathrm{f}})$;

iii) The dynamics of M and L are locally Lipschitz in (y, M) and $(y, \hat{x}, M, L)$, respectively.

Proof Let's first prove Statement (i) by a contradiction argument. Otherwise, there would exist a time $t_{\mathrm{M}} \in [0,\ T_{\mathrm{f}})$, at which $M(t_{\mathrm{M}}) < 1$. In fact, from (7), it can be known that $\dot{M} \geqslant -\beta_1 M^2$ when $M > 0$. This shows that no matter what the system output y is, $M(t) > 0$ has bounded derivative in any time interval on which $\dot{M} \leqslant 0$ and hence is a continuous function in time on this interval. Based on the fact and from $M(0) = 1,\ M(t_{\rm M}) < 1$, it is not hard to see that there exists another time $t'_m \in [0, t_{\rm M})$, at which $M(t'_{\rm M}) = 1,\ \dot{M}(t'_{\rm M}) < 0$, and also a sufficiently small constant $\tau > 0$, such that $\dot{M}(t) < 0$ and $0 < M(t) < 1$ for $\forall t \in (t'_{\rm M},\ t'_{\rm M} + \tau)$. However, (7) and $\dot{M} < 0$ on $(t'_{\rm M},\ t'_{\rm M} + \tau)$ means that for $\forall t \in (t'_{\rm M},\ t'_{\rm M} + \tau), M(t) > \dfrac{\beta_2(1 + |y|^p)^2}{\beta_1} \geqslant \dfrac{\beta_2}{\beta_1} \geqslant 1$, which clearly contradicts the just established $M < 1$ on $(t'_{\rm M},\ t'_{\rm M} + \tau)$, and consequently Statement i) is true.

We next turn to proving the remainder two statements of Proposition 1. First from the statement i) and (7), we know $\dot{L}(t) \geqslant 0,\ \forall t \in [0,\ T_{\rm f})$, and hence $L(t) \geqslant L(0) = 1$, $\forall t \in [0,\ T_{\rm f})$. Statement ii) is thus proven. In Statement iii), the locally Lipschitz property of the dynamics of M is because that it is smooth in both M and $|y|$, and in addition $|y|$ is locally Lipschitz in y. It remains only to verify the locally Lipschitz property of the dynamics of L in $(y, \hat{x}, M, L)$, which is true since noting Statements i) and ii), $\max\{\psi, 0\}$ is locally Lipschitz in ψ and $\dfrac{2(y - \hat{x}_1)^2}{r^{2a}} - \dfrac{\lambda^2}{2r} + 2\sum\limits_{i=1}^{n} \dfrac{\hat{x}_i^2}{r^{2i-2+2a}}$ is a smooth function of $(y, \hat{x}, M, L)$.

Before completely proving the theorem, we next provide two important propositions, which are both rigorously proven in Appendix. Specifically, Proposition 2 characterizes the dynamic behavior of the closed-loop system via a Lyapunov candidate function and will play a central role in the later analysis for stability and tracking performance. Proposition 3 reveals the intrinsic relationship between the high-gain L and the other system states and shows that in order to prove the global boundedness of the closed-loop system, it suffices to prove that of L.

Proposition 2 For the same closed-loop systems as in Theorem 1 with the same design parameters, there exist a known constant $\gamma > 0$, an unknown constant $\Theta > 0$ and symmetric positive definite matrices P and Q, such that on $[0,\ T_{\rm f})$, $V(\varepsilon, z) = \gamma V_1(\varepsilon) + V_2(z) := \gamma \varepsilon^{\rm T} P \varepsilon + \eta^{\rm T} Q \eta$ satisfies

$$\dot{V} \leqslant -(0.5r - \Theta)(\|\varepsilon\|^2 + \|\eta\|^2) + \frac{\Theta}{r}.$$

Proposition 3 For the same closed-loop system as in Theorem 1 with the same $[0,\ T_{\rm f})$, then all the other system states are bounded on $[0,\ T_{\rm f})$ as well.

We can now proceed the main proof of Theorem 1 with the help of the above Propositions 1, 2 and 3. The first claim of the theorem is directly obtained from the following lemma whose proof is provided in Appendix for the sake of compactness though it is the most technical part of the whole proof of the theorem.

Lemma 1 For the same closed-loop system as in Theorem 1 with the same design parameters, $T_{\rm f} = +\infty$ and all the system states are bounded on $[0,\ +\infty)$.

It remains only to prove the global practical tracking can be achieved, i.e., for any prescribed $\lambda > 0$, there exists a finite time T_λ, such that $|y(t)| \leqslant \lambda,\ \forall t \geqslant T_\lambda$.

First, from (5)$-$(7), Proposition 1 and Lemma 1, we can easily verify the following two properties:

a) L is continuously differentiable on $[0,\ +\infty)$ and $\lim\limits_{t\to+\infty} L(t) = \sup\limits_{t\geqslant 0} L(t)$ exists.

b) In the expression of $\dot{L}$, function

$$N(t) = M(\frac{2(y - \hat{x}_1)^2}{r^{2a}} - \frac{\lambda^2}{2r} + 2\sum_{i=1}^{n} \frac{\hat{x}_i^2}{r^{2i-2+2a}})$$

is continuously differentiable in time t and particularly, $\dot{N}$ is global bounded on $[0,\ +\infty)$, i.e., $\sup\limits_{t\geqslant 0} |\dot{N}(t)| < +\infty$.

Then, from property b), it can be concluded the uniform continuity of $\dot{L}$ on $[0, +\infty)$; that is, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, for any $t_1,\ t_2 \in [0, +\infty)$ satisfying $|t_2 - t_1| < \delta$, such that

$$|\dot{L}(t_2) - \dot{L}(t_1)| \leqslant \epsilon.$$

In fact, if δ is chosen such that

$$0 < \delta < \frac{\epsilon}{\max\{1, \sup\limits_{t\geqslant 0} |\dot{N}(t)|\}}$$

for any $\epsilon > 0$, from property a) and the expression of $\dot{L}$ (i.e., (7)), it isn't hard to obtain

$$\begin{aligned} &|\dot{L}(t_2) - \dot{L}(t_1)| \leqslant \\ &|N(t_2) - N(t_1)| \leqslant \sup_{t\geqslant 0} |\dot{N}(t)| \cdot |t_2 - t_1| \leqslant \\ &\frac{\epsilon \sup\limits_{t\geqslant 0} |\dot{N}(t)|}{\max\{1, \sup\limits_{t\geqslant 0} |\dot{N}(t)|\}} \leqslant \epsilon. \end{aligned}$$

Keeping in mind property a) and the uniform continuity of $\dot{L}$, by using Barbălat's Lemma†, we finally establish $\lim\limits_{t\to+\infty} \dot{L}(t) = 0$. Moreover, from Lemma 1, it follows that $\inf\limits_{t\geqslant 0} \dfrac{\lambda^2}{L(t)} > 0$. Therefore, from the expression of $\dot{L}$, we know that for any initial condition of the closed-loop system, there is a finite time $T_\lambda > 0$ such that for any $t > T_\lambda$,

$$\begin{aligned} &M(t)(\frac{2(y(t) - \hat{x}_1(t))^2}{r^{2a}(t)} - \frac{\lambda^2}{2r(t)} + \\ &2\sum_{i=1}^{n} \frac{\hat{x}_i^2(t)}{r^{2i-2+2a}(t)}) \leqslant \inf_{\tau\geqslant 0} \frac{\lambda^2}{2L(\tau)} = \\ &\frac{\lambda^2}{2\sup\limits_{\tau\to+\infty} L(\tau)}, \end{aligned}$$

which together with Proposition 1 implies that for any $t \geqslant T_\lambda$

$$\begin{aligned} &M(t)(\frac{2(y(t) - \hat{x}_1(t))^2}{r^{2a}(t)} - \frac{\lambda^2}{2r(t)} + \\ &2\sum_{i=1}^{n} \frac{\hat{x}_i^2(t)}{r^{2i-2+2a}(t)}) \leqslant \frac{\lambda^2}{2L(t)}, \end{aligned}$$

† **Barbălat's Lemma** Suppose that $\chi : [0, +\infty) \to \mathbb{R}$ is a continuously differentiable function, and $\lim\limits_{t\to+\infty} \chi(t)$ exists and is finite. If $\dot{\chi}(t), t \in [0, +\infty)$ is uniformly continuous, then $\lim\limits_{t\to+\infty} \dot{\chi}(t) = 0$. For more details on the lemma, refer the readers to [26].

and therefore, for any $t > T_\lambda$,

$$M(t)(\frac{2(y(t)-\hat{x}_1(t))^2}{r^{2a}(t)}+ \\ 2\sum_{i=1}^{n}\frac{\hat{x}_i^2(t)}{r^{2i-2+2a}(t)}) \leqslant \frac{\lambda^2}{L(t)}.$$

Thus, by $2a < 1$, Proposition 1 and

$$y^2 \leqslant 2(y-\hat{x}_1)^2 + 2\hat{x}_1^2,$$

we have that for any $t \geqslant T_\lambda$,

$$\frac{\lambda^2}{L(t)} \geqslant \\ M(t)(\frac{2(y(t)-\hat{x}_1(t))^2}{r^{2a}(t)}+\frac{2\hat{x}_1^2(t)}{r^{2a}(t)}) \geqslant \\ M(t)\frac{y^2(t)}{r^{2a}(t)} \geqslant \frac{y^2(t)}{L^{2a}(t)} \geqslant \frac{y^2(t)}{L(t)},$$

which directly concludes that

$$|y(t)| \leqslant \lambda,\ t \geqslant T_\lambda.$$

The proof for the last claim is complete, and hence so is that of the theorem.

Remark 1 Theorem 1 shows that the desired controller is based on the suitably chosen design parameters. Also, from the detailed proof of the theorem, we know that the design parameters should be chosen such that

i) A and B (or l_i's and k_i's) satisfy

$$A^{\mathrm{T}}P+PA \leqslant -I,\ h_1 I \leqslant D_{\mathrm{a}}P+PD_{\mathrm{a}} \leqslant h_2 I, \\ B^{\mathrm{T}}Q+QB \leqslant -2I,\ h_3 I \leqslant D_{\mathrm{a}}Q+QD_{\mathrm{a}} \leqslant h_4 I,$$

where P and Q are symmetric and positive definite, and h_i's are positive constants;

ii) $0 < a < \dfrac{1}{10p}$,

$$0 < \beta_1 \leqslant \frac{1}{\max\{4h_2(1+\|Ql\|^2), 2h_4\}}, \\ \beta_2 \geqslant \max\{\beta_1, \frac{3(1+\|Ql\|^2)}{\min\{h_3, h_1(1+\|Ql\|^2)\}}\}.$$

This is the complete conditions on the design parameters for an appropriate output feedback practical tracking controller.

4 A simulation example

Consider the following second-order uncertain nonlinear system:

$$\begin{cases} \dot{\zeta}_1 = \zeta_2, \\ \dot{\zeta}_2 = u + \theta'\zeta_1^2 \sin\zeta_2|\zeta_2| + \theta', \\ y = \zeta_1 - y_{\mathrm{r}}, \end{cases} \tag{11}$$

where unknown θ' and y_{r} are assumed to be 2 and $\sin(40t)$ on $[0, \infty)$, respectively. It can be verified that this system satisfies the assumptions A1) and A2) with $\theta = 2,\ p = 2$ and $\vartheta = 40$.

As shown in Subsection 2.1, $\hat{x}_1$ and $\hat{x}_2$ are the estimations of $\zeta_1 - y_{\mathrm{r}}$ and ζ_2, respectively, and their dynamics satisfy (6) with $n = 2$. Then, according to (7) and (8), the output-feedback tracking controller of system (11) can be easily designed in the form (8), and furthermore, in virtue of Theorem 1 above, choose the appropriate design parameters as

$$[l_1\ \ l_2]^{\mathrm{T}} = [1\ \ 10]^{\mathrm{T}},\ [k_1\ \ k_2]^{\mathrm{T}} = [12\ \ 1]^{\mathrm{T}}, \\ a = \frac{1}{21},\ \beta_1 = 0.1 \text{ and } \beta_2 = 10.$$

Setting the initial conditions of the closed-loop system by $\zeta_0 = \zeta(0) = [0\ \ 1]^{\mathrm{T}}, \hat{x}(0) = [0\ \ 0]^{\mathrm{T}},\ L(0) = 1$ and $M(0) = 1$, the simulation results are shown in Fig.1−Fig.4. These figures show that all the closed-loop system states, i.e., $\zeta, \hat{x}, M$ and L, are all bounded, and furthermore, demonstrate the effectiveness of the tracking controller designed above, namely, tracking error $|\zeta_1 - \sin(40t)| \leqslant 0.02$ when $t \geqslant 0.15$ s.

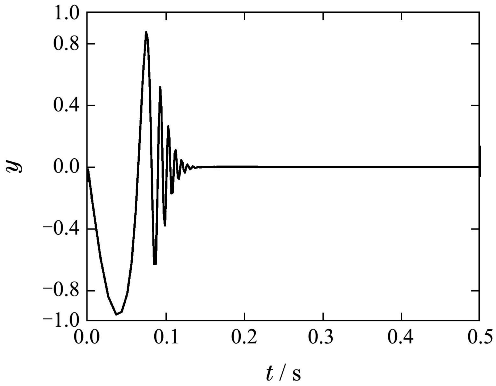

Fig. 1 The trajectory of the tracking error y

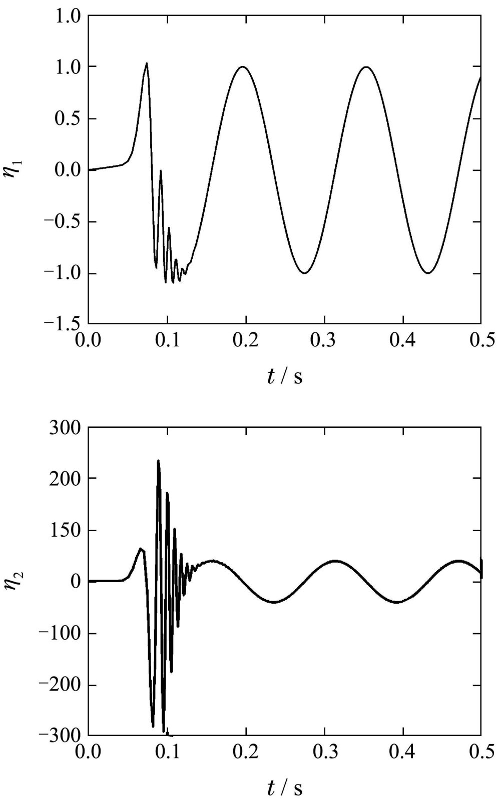

Fig. 2 The trajectories of η_1 and η_2

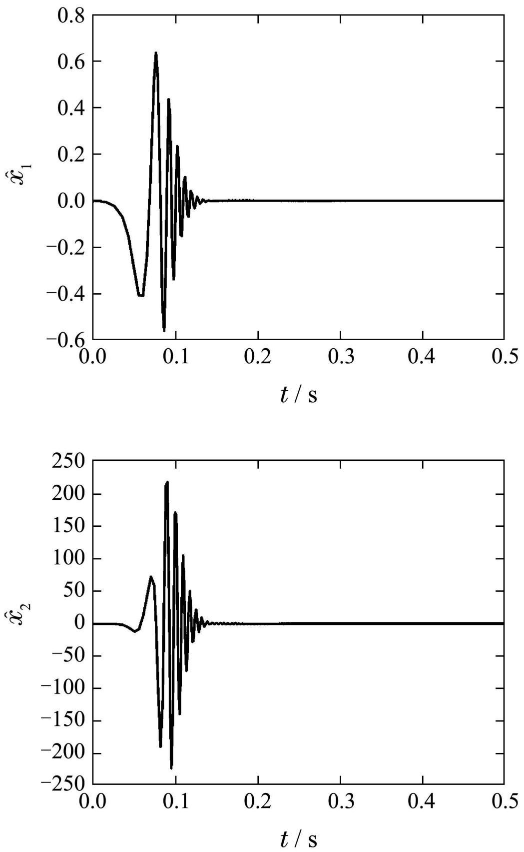

Fig. 3 The trajectories of $\hat{x}_1$ and $\hat{x}_2$

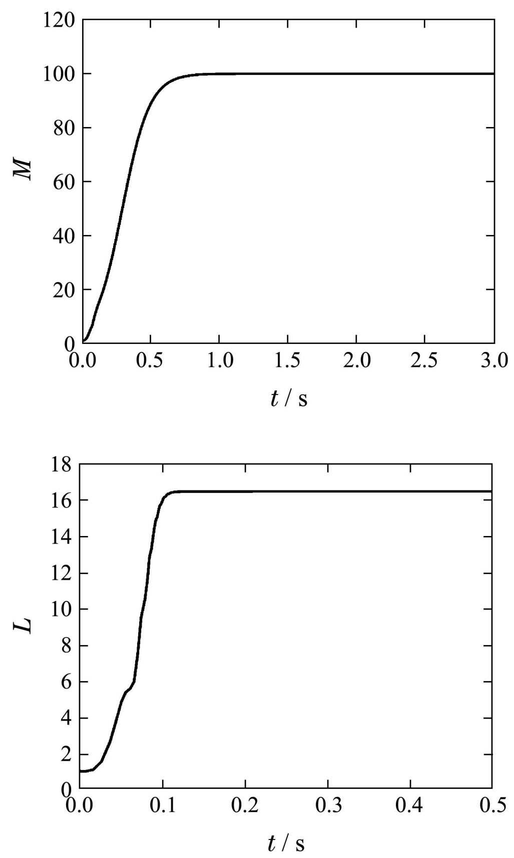

Fig. 4 The trajectories of M and L

5 Concluding remarks

In this paper, the global practical tracking (or λ-tracking) problem has been successfully solved by output-feedback for uncertain nonlinear system (1). Mainly due to the presence of the unmeasurable states dependent growth with the rate of polynomial-of-output of unknown coefficients, the tracking problem of system (1) is rather difficult, and its explicit solution, as the main novelty of the paper, is established by introducing the new dynamic high-gain observer and flexibly combining the ideas and methodologies of universal control and dead zone. It is necessary to point out that the design parameters in the designed controller is undetermined, and their appropriate choice always exits and should be taken within ranges (rather than a set of constants) which would provide control designers with a freedom in choosing the tracking controller. As one can see, the adaptive controller designed is essentially based on the precise knowledge of the upper bound on the highest power of the polynomial-of-output in the growth rate of system (1), and apparently, the current methods are unavailable to the case without such knowledge, for which any tempt deserves special consideration.

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Appendix

The appendix provides the rigorous proofs of fundamental Propositions 2 and 3 and crucial Lemma 1, which is absolutely necessary for the paper.

A1 The detailed proof of Proposition 2

First, the design parameters $l_i,\ i=1,\cdots,n$ and $k_i,\ i=1,\cdots,n$ are chosen to meet Hurwitz condition‡ and such that there exist constants $h_i>0,\ i=1,\cdots,4$ and symmetric positive definite matrices P,Q satisfying

$$\begin{cases} A^{\mathrm T}P+PA\leqslant -I,\ h_1I\leqslant D_{\mathrm a}P+PD_{\mathrm a}\leqslant h_2I,\\ B^{\mathrm T}Q+QB\leqslant -2I,\ h_3I\leqslant D_{\mathrm a}Q+QD_{\mathrm a}\leqslant h_4I. \end{cases} \tag{A.1}$$

It is necessary to point out that the above choice for l_i's and k_i's is always possible according to Lemma 1 of [20].

Let $\gamma=1+\|Ql\|^2$. Then, for $V(\varepsilon,\eta)=\gamma\varepsilon^{\mathrm T}P\varepsilon+\eta^{\mathrm T}Q\eta$, along the trajectories of (10) on $[0,\ T_{\mathrm f})$, we have

$$\begin{aligned}&\dot V\leqslant\\ &-\gamma r\|\varepsilon\|^2+2\gamma\varepsilon^{\mathrm T}Pf-\frac{\gamma\dot r}{r}\varepsilon^{\mathrm T}(D_{\mathrm a}P+PD_{\mathrm a})\varepsilon-\\ &2r\|\eta\|^2+2r\eta^{\mathrm T}Ql\varepsilon_1-\frac{\dot r}{r}\eta^{\mathrm T}(D_{\mathrm a}Q+QD_{\mathrm a})\eta.\end{aligned} \tag{A.2}$$

We next deal with the destabilized terms on the right-hand side of the above inequality. From (7) and Proposition 1, we know $\frac{\dot r}{r}=\frac{\dot M}{M}+\frac{\dot L}{L}\geqslant\frac{\dot M}{M}=-\beta_1M+\beta_2(1+|y|^p)^2$ on $[0,\ T_{\mathrm f})$. Then, from (A.1) and Proposition 1, we have

$$\begin{cases} -\frac{\gamma\dot r}{r}\varepsilon^{\mathrm T}(D_{\mathrm a}P+PD_{\mathrm a})\varepsilon\leqslant\\ \gamma(\beta_1M-\beta_2(1+|y|^p)^2)\varepsilon^{\mathrm T}(D_{\mathrm a}P+PD_{\mathrm a})\varepsilon\leqslant\\ h_2\gamma\beta_1M\|\varepsilon\|^2-h_1\gamma\beta_2(1+|y|^p)^2\|\varepsilon\|^2,\\ -\frac{\dot r}{r}\eta^{\mathrm T}(D_{\mathrm a}Q+QD_{\mathrm a})\eta\leqslant\\ (\beta_1M-\beta_2(1+|y|^p)^2)\eta^{\mathrm T}(D_{\mathrm a}Q+QD_{\mathrm a})\eta\leqslant\\ h_4\beta_1M\|\eta\|^2-h_3\beta_2(1+|y|^p)^2\|\eta\|^2. \end{cases} \tag{A.3}$$

We then handle the second and the fifth terms on the right-hand side of (A.2). By (4) and (9), Proposition 1, we know that

$$|y|\leqslant|\tilde x_1|+|\hat x_1|\leqslant r^a|\varepsilon_1|+r^a|\eta_1|\leqslant r^a(\|\varepsilon\|+\|\eta\|).$$

Thus, by (3) and the inequalities that $|y|^p\leqslant\frac{p}{p+1}|y|^{p+1}+\frac{1}{p+1}$ and $|y|^{p+1}\leqslant(1+|y|^p)|y|$ for any $y\in\mathbb R$, we have

$$\begin{aligned}&|f_1|=|\frac{\phi_1-\dot y_{\mathrm r}}{r^a}|\leqslant\frac{|\phi_1|+|\dot y_{\mathrm r}|}{r^a}\leqslant\\ &\frac{\bar\theta(1+|y|^p)|\zeta_1|}{r^a}+\frac{\theta+\vartheta}{r^a}\leqslant\\ &\frac{\bar\theta(1+|y|^p)(|y|+\vartheta)}{r^a}+\frac{\theta+\vartheta}{r^a}\leqslant\\ &\frac{\bar\theta}{r^a}((1+\vartheta)(1+|y|^p)|y|+\frac{\vartheta(p+2)}{p+1})+\frac{\theta+\vartheta}{r^a}\leqslant\\ &(1+\vartheta)\bar\theta(1+|y|^p)(\|\varepsilon\|+\|\eta\|)+\\ &\frac{\bar\theta\vartheta(1+\frac{1}{p+1})}{r^a}+\frac{\theta+\vartheta}{r^a},\end{aligned}$$

where

$$\bar\theta=\theta\max\{(1+2^{p-1}\vartheta^p),2^{p-1}\},$$

and for $i=2,\cdots,n$,

$$\begin{aligned}&|f_i|=\Big|\frac{\phi_i}{r^{a+i-1}}\Big|\leqslant\\ &\frac{\bar\theta(1+|y|^p)(\vartheta+\sum\limits_{j=1}^{i}(|\tilde x_j|+|\hat x_j|))+\theta}{r^{a+i-1}}\leqslant\end{aligned}$$

‡That is, polynomials $s^n+l_1s^{n-1}+\cdots+l_{n-1}s+l_n$ and $s^n+k_1s^{n-1}+\cdots+k_{n-1}s+k_n$ are Hurwitz.

$$\frac{\bar{\theta}(1+|y|^p)(\vartheta+\sum\limits_{j=1}^{i} r^{a+j-1}(|\varepsilon_j|+|\eta_j|))+\theta}{r^{a+i-1}}\leqslant$$
$$\bar{\theta}((1+|y|^p)\sum_{j=1}^{i}(|\varepsilon_j|+|\eta_j|)+\tfrac{\vartheta(1+|y|^p)}{r})+\tfrac{\theta}{r^a}\leqslant$$
$$\bar{\theta}(\sqrt{i}(1+|y|^p)(\|\varepsilon\|+\|\eta\|)+\frac{\vartheta(1+|y|^p)}{r})+\frac{\theta}{r^a}.$$

Combing the above estimations results in

$$\|f\|_\infty=\max_{i=1,\cdots,n}|f_i|\leqslant$$
$$\bar{\Theta}(1+|y|^p)(\|\varepsilon\|+\|\eta\|)+\frac{\bar{\Theta}(1+|y|^p)}{r}+\frac{\bar{\Theta}}{r^a},$$

where $\bar{\Theta}=\theta+\vartheta+\sqrt{n}(1+2\vartheta)\bar{\theta}$ is obviously an unknown positive constant.

Therefore, by the method of completing square, the second term on the right-hand side of (A.2) satisfies (noting that $1-2a>0$ and $r(t)>1,\ \forall t\in\mathbb{R}^+$)

$$\begin{aligned}&2\gamma\varepsilon^{\mathrm{T}}Pf\leqslant 2\gamma\|P\|\cdot\|\varepsilon\|\cdot\|f\|_\infty\leqslant\\&2\gamma\|P\|\cdot\|\varepsilon\|(\bar{\Theta}(1+|y|^p)(\|\varepsilon\|+\|\eta\|)+\\&\frac{\bar{\Theta}(1+|y|^p)}{r}+\frac{\bar{\Theta}}{r^a})\leqslant\\&\gamma\bar{\Theta}^2\|P\|^2\|\varepsilon\|^2+2\gamma(1+|y|^p)^2(\|\varepsilon\|^2+\|\eta\|^2)+\\&\gamma(1+|y|^p)^2\|\varepsilon\|^2+\frac{\gamma\bar{\Theta}^2\|P\|^2}{r^2}+\frac{r^{1-2a}\|\varepsilon\|^2}{4}+\\&\frac{4\gamma^2\bar{\Theta}^2\|P\|^2}{r}\leqslant\\&(\frac{r^{1-2a}}{4}+\gamma\bar{\Theta}^2\|P\|^2)\|\varepsilon\|^2+\\&3\gamma(1+|y|^p)^2(\|\varepsilon\|^2+\|\eta\|^2)+\frac{D_1(\bar{\Theta})}{r},\end{aligned}\tag{A.4}$$

where $D_1=5\gamma^2\bar{\Theta}^2\|P\|^2$ is an unknown positive constant.

In addition, the fifth term on the right-hand side of (A.2) satisfies

$$\begin{aligned}&2r\eta^{\mathrm{T}}Ql\varepsilon_1\leqslant\\&2r\|Ql\|\cdot\|\eta\|\cdot|\varepsilon_1|\leqslant r\|Ql\|^2\|\varepsilon\|^2+r\|\eta\|^2.\end{aligned}$$

Substituting this, (A.3) and (A.4) into (A.2), we obtain

$$\begin{aligned}\dot{V}\leqslant&-(\gamma r-r\|Ql\|^2-\frac{r^{1-2a}}{4}-\\&h_2\gamma\beta_1M-\gamma\bar{\Theta}^2\|P\|^2)\|\varepsilon\|^2-(L-h_4\beta_1)M\|\eta\|^2-\\&(h_3\beta_2-3\gamma)(1+|y|^p)^2\|\eta\|^2-\\&(h_1\gamma\beta_2-3\gamma)(1+|y|^p)^2\|\varepsilon\|^2+\frac{D_1(\bar{\Theta})}{r},\end{aligned}\tag{A.5}$$

where the design parameters a,β_1 and β_2 are chosen as follows:

$$\begin{cases}0<a<\dfrac{1}{10p},\\0<\beta_1\leqslant\dfrac{1}{\max\{4\gamma h_2,2h_4\}},\\\beta_2\geqslant\max\{\beta_1,\dfrac{3\gamma}{\min\{h_3,h_1\gamma\}}\}.\end{cases}\tag{A.6}$$

It is clear that inequalities of (A.6) have absorbed the preceding constraints on a,β_1 and β_2 given in subsection 2.2, i.e., $0<a<\dfrac{1}{10p}$ and $0<\beta_1\leqslant\beta_2$. Moreover, according to (A.1), one can see that h_i's depend on a, so do β_1 and β_2, and hence the choice described by (A.6) should be implemented in the order of first a, then β_1, and finally β_2.

From (A.6) and the facts $p\geqslant1,\ r=LM,\ L(t)\geqslant1,\ M(t)\geqslant1,\ \forall t\in[0,+\infty)$, it clearly follows that

$$\begin{cases}0<1-2a<1,\\\dfrac{r^{1-2a}}{4}+h_2\gamma\beta_1M\leqslant0.5r,\\(L-h_4\beta_1)M\geqslant0.5r,\\h_3\beta_2-3\gamma\geqslant0,\\h_1\gamma\beta_2-3\gamma\geqslant0,\end{cases}$$

and therefore from (A.5) and noting $\gamma=1+\|Ql\|^2>1$, we have

$$\dot{V}\leqslant-\big(0.5r-\Theta\big)(\|\varepsilon\|^2+\|\eta\|^2)+\frac{\Theta}{r},\tag{A.7}$$

where $\Theta=\max\{\gamma\bar{\Theta}^2\|P\|^2,\ D_1(\bar{\Theta})\}$. This completes the proof of Proposition 2.

A2 The detailed proof of Proposition 3

By the preceding definitions of ε,η and r, it is easy to know that the boundedness of x and $\hat{x}$ can be implied by that of η,ε, M and L. Therefore, to complete the proof, it suffices to prove the boundedness of η, ε and M, based on that of L on $[0,\ T_{\mathrm{f}})$.

Let's first show that η is bounded on $[0,\ T_{\mathrm{f}})$. Consider the function $V_\eta(\eta)=\eta^{\mathrm{T}}Q\eta$ for (10). Then, by (A.1), $\dfrac{\dot{r}}{r}=\dfrac{\dot{M}}{M}+\dfrac{\dot{L}}{L}\geqslant\dfrac{\dot{M}}{M}$ and the foregoing updating law of M, we have

$$\begin{aligned}&\dot{V}_\eta=\\&-2r\|\eta\|^2+2r\eta^{\mathrm{T}}Ql\varepsilon_1-\frac{\dot{r}}{r}\eta^{\mathrm{T}}(D_{\mathrm{a}}Q+QD_{\mathrm{a}})\eta\leqslant\\&-2r\|\eta\|^2+2r\eta^{\mathrm{T}}Ql\varepsilon_1-\frac{\dot{M}}{M}\eta^{\mathrm{T}}(D_{\mathrm{a}}Q+QD_{\mathrm{a}})\eta\leqslant\\&-2r\|\eta\|^2+2r\eta^{\mathrm{T}}Ql\varepsilon_1+h_4\beta_1M\|\eta\|^2-\\&h_3\beta_2(1+|y|^p)^2\|\eta\|^2.\end{aligned}\tag{A.8}$$

Noticing

$$2r\eta^{\mathrm{T}}Ql\varepsilon_1\leqslant\frac{1}{2}r\|\eta\|^2+2r\|Ql\|^2\varepsilon_1^2,$$

the facts $h_4\beta_1\leqslant\dfrac{1}{2},\ h_3\beta_2>0$ (ensured by (A.6)), the supposed boundedness of L on $[0,+\infty)$, and using (7) and (A.8), we have

$$\begin{aligned}&\dot{V}_\eta\leqslant\\&-r\|\eta\|^2+2r\|Ql\|^2\varepsilon_1^2-h_3\beta_2(1+|y|^p)^2\|\eta\|^2\leqslant\\&-r\|\eta\|^2+\|Ql\|^2LM(2\varepsilon_1^2+2\|\eta\|^2-\frac{\lambda^2}{2r})+\\&\frac{\lambda^2\|Ql\|^2}{2}\leqslant-c_1rV_\eta+\|Ql\|^2L(T_{\mathrm{f}})\dot{L}+\frac{\lambda^2\|Ql\|^2}{2},\end{aligned}$$

where $c_1=\dfrac{1}{\lambda_{\max}(Q)}>0$, and $L(T_{\mathrm{f}})$ denotes the maximum of L on $[0,\ T_{\mathrm{f}})$. From this, it is easy to see that on $[0,\ T_{\mathrm{f}})$ (keeping in mind the fact $r\geqslant1$),

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_\eta(\eta(t))\mathrm{e}^{c_1t})\leqslant\|Ql\|^2L(T_{\mathrm{f}})\dot{L}(t)\mathrm{e}^{c_1t}+\frac{\lambda^2\|Ql\|^2e^{c_1t}}{2}.\tag{A.9}$$

Then, from 0 to any $t\in[0,\ T_{\mathrm{f}})$, integrating both sides of (A.9) directly yields

$$V_\eta(\eta(t))\mathrm{e}^{c_1t}\leqslant$$

$$V_\eta(\eta(0)) + \|Ql\|^2 L(T_{\rm f}) \int_0^t {\rm e}^{c_1\tau} {\rm d}L + \frac{\lambda^2\|Ql\|^2}{2}\int_0^t {\rm e}^{c_1\tau}{\rm d}\tau = V_\eta(\eta(0)) + \|Ql\|^2 L(T_{\rm f})({\rm e}^{c_1 t}L(t) - L(0) - c_1\int_0^t L(\tau){\rm e}^{c_1\tau}{\rm d}\tau) + \frac{\lambda^2\|Ql\|^2}{2c_1}({\rm e}^{c_1 t}-1) \leqslant V_\eta(\eta(0)) + \|Ql\|^2 L^2(T_{\rm f})e^{c_1 t} + \frac{\lambda^2\|Ql\|^2}{2c_1}e^{c_1 t},$$

which shows that for $\forall t \in [0,\ T_{\rm f})$,

$$V_\eta(\eta(t)) \leqslant V_\eta(\eta(0))e^{-c_1 t} + \|Ql\|^2 L^2(T_{\rm f}) + \frac{\lambda^2\|Ql\|^2}{2c_1},$$

and hence η is bounded on $[0,\ T_{\rm f})$.

We next show that $\tilde{x}$ is bounded on $[0,\ T_{\rm f})$ as well. For this purpose, we introduce the following scaling transformation (similar to that in [21]):

$$\xi_i = \frac{\tilde{x}_i}{(L^*M)^{a+i-1}},\ i = 1,\cdots,n,$$

where $L^* \geqslant L(T_{\rm f})$ is a positive constant to be chosen large enough. Then, in terms of the foregoing dynamics of $\tilde{x}$ and M, we have

$$\begin{cases} \dot{\xi}_i = L^*M\xi_{i+1} - L^*Ml_i\xi_1 + L^*Ml_i\xi_1 - LM(\dfrac{L}{L^*})^{i-1}l_i\xi_1 + \dfrac{\varphi_i(\cdot)}{(L^*M)^{i+a-1}} - \dfrac{\dot{M}}{M}(i+a-1)\xi_i,\ i = 1,\cdots,n-1, \\ \dot{\xi}_n = -L^*Ml_n\xi_1 + L^*Ml_n\xi_1 - LM(\dfrac{L}{L^*})^{n-1}l_n\xi_1 + \dfrac{\varphi_n(\cdot)}{(L^*M)^{n+a-1}} - \dfrac{\dot{M}}{M}(n+a-1)\xi_n, \end{cases}$$

which can be compactly rewritten in the following form:

$$\dot{\xi} = L^*MA\xi + L^*Ml\xi_1 - LM\Gamma l\xi_1 + f^*(t,x,u) - \frac{\dot{M}}{M}D_{\rm a}\xi, \tag{A.10}$$

where

$$\xi = [\xi_1\ \cdots\ \xi_n]^{\rm T},$$
$$\Gamma = {\rm diag}\{1, \frac{L}{L^*},\cdots,(\frac{L}{L^*})^{n-1}\},$$

A, l and $D_{\rm a}$ are the same as before, and

$$f^* = [f_1^*\ \cdots\ f_n^*]^{\rm T} = [\frac{\varphi_1(\cdot)}{(L^*M)^a}\ \frac{\varphi_2(\cdot)}{(L^*M)^{a+1}}\ \cdots\ \frac{\varphi_n(\cdot)}{(L^*M)^{a+n-1}}]^{\rm T}. \tag{A.11}$$

Then, along the solutions of (A.10), the Lyapunov candidate function $V_\xi(\xi) = \xi^{\rm T}P\xi$ satisfies

$$\dot{V}_\xi \leqslant -L^*M\|\xi\|^2 + 2L^*M\xi^{\rm T}Pl\xi_1 - 2LM\xi^{\rm T}P\Gamma l\xi_1 + 2\xi^{\rm T}Pf^* - \frac{\dot{M}}{M}\xi^{\rm T}(D_{\rm a}P + PD_{\rm a})\xi. \tag{A.12}$$

Let's handle the destabilized terms on the right-hand side of the above inequality. By the method of completing square, we obtain

$$\begin{cases} 2L^*M\xi^{\rm T}Pl\xi_1 \leqslant 4L^*M\|Pl\|^2\xi_1^2 + \dfrac{L^*M\|\xi\|^2}{4}, \\ 2LM\xi^{\rm T}P\Gamma l\xi_1 \leqslant 4LM\|P\Gamma l\|^2\xi_1^2 + \dfrac{LM\|\xi\|^2}{4}. \end{cases} \tag{A.13}$$

Moreover, by (3)–(4)(A.11) and the fact $L^*M \geqslant 1$ on $[0,\ T_{\rm f})$, we have

$$|f_1^*| = |\frac{\phi_1 - \dot{y}_{\rm r}}{(L^*M)^a}| \leqslant \frac{|\phi_1| + |\dot{y}_{\rm r}|}{(L^*M)^a} \leqslant \frac{\bar{\theta}(1+|y|^p)|x_1|}{(L^*M)^a} + \frac{\vartheta}{(L^*M)^a} \leqslant \frac{\bar{\theta}(1+|y|^p)(|\tilde{x}_1| + |\hat{x}_1|)}{(L^*M)^a} + \frac{\theta+\vartheta}{(L^*M)^a} \leqslant \bar{\theta}(1+|y|^p)(|\xi_1| + |\eta_1|) + \frac{\theta+\vartheta}{(L^*M)^a} \leqslant \bar{\theta}(1+|y|^p)(\|\varepsilon\| + \|\eta\|) + \frac{\theta+\vartheta}{(L^*M)^a},$$

and similarly for $i = 2,\cdots,n$,

$$|f_i^*| = |\frac{\phi_i}{(L^*M)^{a+i-1}}| \leqslant \frac{\bar{\theta}(1+|y|^p)(\vartheta + \sum\limits_{j=1}^{i}(|\tilde{x}_j| + |\hat{x}_j|)) + \theta}{(L^*M)^{a+i-1}} \leqslant \bar{\theta}(1+|y|^p)(\sum_{j=1}^{i}(|\xi_j| + |\eta_j|) + \frac{\vartheta}{L^*M}) + \frac{\theta}{(L^*M)^a} \leqslant \bar{\theta}(1+|y|^p)(\sqrt{i}(\|\xi\| + \|\eta\|) + \frac{\vartheta}{L^*M}) + \frac{\theta}{(L^*M)^a},$$

which together with the preceding definition of unknown constant $\bar{\Theta}$ result in

$$\|f^*\|_\infty = \max_{i=1,\cdots,n}|f_i^*| \leqslant \bar{\Theta}(1+|y|^p)(\|\xi\| + \|\eta\|) + \frac{\bar{\Theta}(1+|y|^p)}{L^*M} + \frac{\bar{\Theta}}{(L^*M)^a}.$$

Based on above estimation and by the method of completing square, we have

$$2\xi^{\rm T}Pf^* \leqslant 2\|P\|\cdot\|\xi\|(\bar{\Theta}(1+|y|^p)(\|\xi\| + \|\eta\|) + \frac{\bar{\Theta}(1+|y|^p)}{L^*M} + \frac{\bar{\Theta}}{(L^*M)^a}) \leqslant \frac{h_1\beta_2}{2M}(1+|y|^p)^2\|\xi\|^2 + \frac{4}{h_1\beta_2}\bar{\Theta}^2\|P\|^2M\cdot(\|\xi\|^2 + \|\eta\|^2) + \frac{h_1\beta_2}{2L^*M}(1+|y|^p)^2\|\xi\|^2 + \frac{2\bar{\Theta}^2\|P\|^2}{L^*Mh_1\beta_2} + \frac{\bar{\Theta}\|P\|}{(L^*M)^a}(1+\|\xi\|^2) \leqslant MD_2(\bar{\Theta})(\|\xi\|^2 + \|\eta\|^2) + \frac{h_1\beta_2}{M}(1+|y|^p)^2\|\xi\|^2 + D_2(\bar{\Theta}),$$

where $D_2 = \dfrac{4}{h_1\beta_2}\bar{\Theta}^2\|P\|^2 + \bar{\Theta}\|P\|$ is an unknown positive constant. From this together with (A.12)–(A.13) and the updating law of M, it follows that

$$\dot{V}_\xi \leqslant -L^*M\|\xi\|^2 + 4L^*M\|Pl\|^2\xi_1^2 + \frac{L^*M\|\xi\|^2}{4} + 4LM\|P\Gamma l\|^2\xi_1^2 + \frac{LM\|\xi\|^2}{4} + MD_2(\bar{\Theta})(\|\xi\|^2 + \|\eta\|^2) + \frac{h_1\beta_2}{M}(1+|y|^p)^2\|\xi\|^2 + D_2(\bar{\Theta}) - \big(-\beta_1M + \beta_2(1+|y|^2)^2\big)\xi^{\rm T}\big(D_{\rm a}P + PD_{\rm a}\big)\xi \leqslant -(0.5L^* - D_2(\bar{\Theta}) - h_2\beta_1)M\|\xi\|^2 + MD_2(\bar{\Theta})\|\eta\|^2 +$$

$4M(L^*\|Pl\|^2 + L\|P\Gamma l\|^2)\xi_1^2 - h_1\beta_2(1-M^{-1})(1+|y|^p)^2\|\xi\|^2 + D_2(\bar\Theta)$.

If L^* is chosen large enough such that

$$L^* \geqslant \max\{L(T_{\rm f}),\ 4D_2(\bar\Theta)+4h_2\beta_1\},$$

and noting the foregoing updating law of L and $M(t)\geqslant 1,\ t\in\mathbb{R}^+$, we have

$$\begin{aligned}\dot V_\xi \leqslant\ & -0.25L^*M\|\xi\|^2 + MD_2(\bar\Theta)\|\eta\|^2 + \\ & 4M(L^*\|Pl\|^2 + L\|P\Gamma l\|^2)\xi_1^2 + D_2(\bar\Theta) \leqslant \\ & -0.25L^*M\|\xi\|^2 + (0.5D_2(\bar\Theta) + \\ & 2(L^*\|Pl\|^2 + L(T_{\rm f})\|P\Gamma l\|^2))\dot L + \\ & \frac{\lambda^2\left(0.5D_2(\Theta) + 2(L^*\|Pl\|^2 + L(T_{\rm f})\|P\Gamma l\|^2)\right)}{2} + \\ & D_2(\bar\Theta),\end{aligned}$$

from which, noting the boundedness of $\|P\Gamma l\|$ by the foregoing definition of Γ and applying the same reasoning as the above proof of the boundedness of η, we can easily show that ξ is bounded on $[0,\ T_{\rm f})$, as well as ε.

It remains only to prove the boundedness of M on $[0,\ T_{\rm f})$. In fact, from the boundedness of L, η and ε on $[0,\ T_{\rm f})$, we easily know that

$$|y(t)| \leqslant DM^a(t),\ \forall t\in[0, T_{\rm f}),$$

where positive constant $D = L(t_{\rm f})\sup\limits_{t\in[0,t_{\rm f})}\big(|\varepsilon_1(t)| + |\eta_1(t)|\big)$.

Then, from the updating law of M, it follows that on $[0,\ T_{\rm f})$

$$\begin{aligned}\dot M &= -\beta_1M^2 + \beta_2(1+|y|^p)^2M \leqslant \\ &-\beta_1M^2 + 2D^{2ap}M^{1+2ap} + 2\beta_2M,\end{aligned}$$

which together with the fact $1+2ap<2$ directly concludes the boundedness of M on $[0,\ T_{\rm f})$.

So far, the proof of Proposition 3 is complete.

A3 The detailed proof of Lemma 1

To begin with, let's prove $T_{\rm f}=+\infty$ by a contradiction argument. Suppose that $T_{\rm f}$ is finite. Then, to seek contradictions, consider the following two mutually exclusive cases:

Case 1 L is bounded on $[0,\ T_{\rm f})$.

Case 2 L is unbounded on $[0,\ T_{\rm f})$ i.e., L escapes at $T_{\rm f}$.

First for Case 1, because $T_{\rm f}$ is finite and as pointed out before, $[0,\ T_{\rm f})$ is the maximal interval of existence of the solution of the closed-loop system, we know there must exist at least one other closed-loop system state which escapes at $T_{\rm f}$. However, by Proposition 3, the boundedness of L on $[0,\ T_{\rm f})$ implies that ε, η and M are bounded on $[0,\ t_{\rm f})$, and so are all the closed-loop system states. This obviously results in a contradiction and consequently shows Case 1 does not occur.

Second for Case 2, since the solution of the closed-loop system is well defined and L is monotone nondecreasing, both on $[0,\ T_{\rm f})$, there must exist a finite time $t^*\in(0,\ T_{\rm f})$, such that

$$L(t) \geqslant 2+2\Theta,\ \forall t\in[t^*,\ T_{\rm f}),$$

which and (A.7) result in

$$\begin{aligned}\dot V \leqslant\ & -M(\|\varepsilon\|^2+\|\eta\|^2)+\Theta \leqslant \\ & -c_2MV+\Theta,\quad \forall t\in[t^*,\ T_{\rm f}),\end{aligned}\tag{A.14}$$

where $c_2 = \dfrac{1}{\max\{\gamma\lambda_{\max}(P),\ \lambda_{\max}(Q)\}}$. From this, it is easy to show that ε and η are bounded on $[0,\ T_{\rm f})$, and furthermore, by the finiteness of $T_{\rm f}$,

$$\int_{t^*}^{T_{\rm f}} M(t)V(t){\rm d}t + \int_0^{t^*} M(t)V(t){\rm d}t \leqslant \frac{1}{c_2}(V(t^*)+\Theta T_{\rm f}) + \int_0^{t^*} M(t)V(t){\rm d}t < +\infty.$$

Then, after integrating $\dot L = M\max\{2\varepsilon_1^2 + 2\|\eta\|^2 - (2r)^{-1}\lambda^2,\ 0\}$ from 0 to $T_{\rm f}$, we have

$$\begin{aligned}&+\infty = L(T_{\rm f}) - L(0) = \\ &\int_0^{T_{\rm f}} M(t)\max\{2\varepsilon_1^2(t) + 2\|\eta(t)\|^2 - \frac{\lambda^2}{2r(t)},\ 0\}{\rm d}t \leqslant \\ &\int_0^{T_{\rm f}} M(t)(2\varepsilon_1^2(t) + 2\|\eta(t)\|^2){\rm d}t \leqslant \\ &\int_0^{T_{\rm f}} c_3M(t)V(t){\rm d}t < +\infty,\end{aligned}$$

where $c_3 = \dfrac{2}{\min\{\gamma\lambda_{\min}(P),\ \lambda_{\min}(Q)\}}$, which is a contradiction and hence shows that Case 2 is impossible to occur.

So far, the infiniteness of $T_{\rm f}$ is concluded from the above two contradictions brought out in Cases 1 and 2.

We next turn to proving the boundedness of $x, \hat x, L$ and M on $[0,\ T_{\rm f}) = [0,\ +\infty)$. As discussed before, it suffices to prove the boundedness of ε, η L and M on $[0,\ +\infty)$. Furthermore, by Proposition 3, we only need to show the boundedness of L on $[0,\ +\infty)$. This will be proceeded by a contradiction argument. Suppose that L is unbounded on $[0,\ +\infty)$. Then, by the monotone nondecreasing property of L, we know that $\lim\limits_{t\to+\infty} L(t) = +\infty$. Thus, due to the continuity of L on $[0,\ +\infty)$, for any small constant $\sigma\in(0,\ 1)$, there exists a finite time $T_\sigma\in(0, +\infty)$, at which

$$L(T_\sigma) = 1 + 2\Theta + \frac{2}{\sigma c_2},\tag{A.15}$$

where c_2 is the same constant as defined in (A.14). By this, (A.7) and the fact $L(t)\geqslant L(T_\sigma),\ \forall t\in[T_\sigma,\ +\infty)$, we know that on $[T_\sigma,\ +\infty)$,

$$\begin{aligned}\dot V \leqslant\ & -\frac{1}{\sigma c_2}(\|\varepsilon\|^2 + \|z\|^2) + \frac{\Theta}{r(T_\sigma)} \leqslant \\ & -\frac{1}{\sigma}V + \frac{\Theta}{L(T_\sigma)}.\end{aligned}\tag{A.16}$$

Multiplying ${\rm e}^{\frac{1}{\sigma}t}$ on both sides of (A.16), it is easy to see

$$\frac{\rm d}{{\rm d}t}({\rm e}^{\frac{1}{\sigma}t}V(t)) \leqslant \frac{\Theta}{L(T_\sigma)}{\rm e}^{\frac{1}{\sigma}t},\ \forall t\in[T_\sigma,\ +\infty),$$

where and in what follows, $V(t)$ denotes $V(\varepsilon(t),\ \eta(t))$ if no confusion occurs. After integrating both sides of the above inequality from T_σ to t, we yield

$$V(t) \leqslant \frac{\sigma\Theta}{L(T_\sigma)} + {\rm e}^{\frac{1}{\sigma}(T_\sigma - t)}V(T_\sigma),\tag{A.17}$$

for $\forall\, t\in[T_\sigma,\ +\infty)$.

It is noticed that (A.15) means that T_σ is monotone increasing to infinity as σ goes to zero. Thus, there holds that $T_\sigma \geqslant T_1$ since $\sigma\leqslant 1$, where $T_1 = T_\sigma|_{\sigma=1}$. This and (A.17) imply that

$$\begin{aligned}V(T_\sigma) \leqslant\ & \frac{\sigma\Theta}{L(T_1)} + e^{\frac{1}{\sigma}(T_1 - T_\sigma)}V(T_1) \leqslant \\ & \Theta + V(T_1),\end{aligned}$$

from which and (A.17), if follows that $V(t)\leqslant 2\Theta + V(T_1)$, $\forall\, t\in[T_\sigma,\ +\infty)$, and consequently,

$$\begin{aligned}&V(t) \leqslant \max\{2\Theta + V(T_1),\ \sup_{0\leqslant\tau\leqslant T_1} V(\tau)\} =: \bar V, \\ &\qquad \forall\, t\in[0,\ +\infty).\end{aligned}\tag{A.18}$$

By (A.18) and the preceding definitions of V and c_3, we easily have $|y| \leqslant r^a(c_3\bar{V})^{\frac{1}{2}}$ on $[0, +\infty)$. Then from Proposition 1 and the foregoing updating law of M, it follows that

$$\begin{aligned}\dot{M} =& -\beta_1 M^2 + \beta_2(1+|y|^p)^2 M \leqslant \\ & -\beta_1 M^2 + 2\beta_2\left(1 + r^{2ap}c_3^p\bar{V}^p\right)M \leqslant \\ & -\beta_1 M^2 + 2\beta_2(1 + L^{2ap}c_3^p\bar{V}^p)M^{1+2ap}.\end{aligned}$$

This together with $M(0) = 1$ and $0 < \beta_1 \leqslant \beta_2$ concludes that

$$M(t) \leqslant (\frac{2\beta_2}{\beta_1}(1 + c_3^p\bar{V}^p(\max_{0\leqslant\tau\leqslant t} L(\tau))^{2ap}))^{\frac{1}{1-2ap}},$$

for $\forall t \in [0, +\infty)$, which and the monotone nondecreasing property of L on $[0, +\infty)$ imply that

$$M(t) \leqslant (2\beta_1^{-1}\beta_2(1 + L^{2ap}(t)c_3^p\bar{V}^p))^{\frac{1}{1-2ap}}, \quad \text{(A.19)}$$

for $\forall t \in [0, +\infty)$. This reveals a quite essential relationship between M and L to be used in the subsequent proof. Also, noting that a is chosen such that $0 < a \leqslant \dfrac{1}{10p}$ and hence $0 < \dfrac{2ap}{1-2ap} \leqslant \dfrac{1}{4}$, we see from (A.15) and (A.19) that

$$M(T_\sigma) \leqslant c_4\sigma^{-\frac{2ap}{1-2ap}} \leqslant c_4\sigma^{-\frac{1}{4}}, \ \forall\sigma \in (0, 1), \quad \text{(A.20)}$$

where

$$c_4 = (2\beta_1^{-1}\beta_2(1 + (1 + 2\Theta + 2/c_2)^{2ap}c_3^p\bar{V}^p))^{\frac{1}{1-2ap}}.$$

Moreover, by (7)(A.18) and the above definitions of c_3 and $\bar{V}$, one has on $[0, +\infty)$

$$\begin{aligned}\dot{L} =& M\max\{2\varepsilon_1^2 + 2\|\eta\|^2 - (2r)^{-1}\lambda^2, 0\} \leqslant \\ & 2M(\varepsilon_1^2 + \|\eta\|^2) \leqslant c_3M\bar{V}.\end{aligned} \quad \text{(A.21)}$$

Also, from (A.19), the facts of $L \geqslant 1$ on $[0, +\infty)$ and $\dfrac{2ap}{1-2ap} \leqslant \dfrac{1}{4}$, it is not hard to obtain for positive constant $c_5 = (2\beta_1^{-1}\beta_2(1 + c_3^p\bar{V}^p))^{\frac{1}{1-2ap}}$,

$$M(t) \leqslant c_5\sqrt[4]{L(t)}, \ \forall t \in [0, +\infty), \quad \text{(A.22)}$$

which together with (A.21) straightforwardly leads to

$$\dot{L}(t) \leqslant c_6\sqrt[4]{L(t)}, \ \forall t \in [0, +\infty),$$

for positive constant $c_6 = c_3c_5\bar{V}$, and in turn

$$L(t) \leqslant (\frac{5c_6}{4}(t - T_\sigma) + L^{\frac{5}{4}}(T_\sigma))^{\frac{4}{5}}, \quad \text{(A.23)}$$

for $\forall t \in [T_\sigma, +\infty)$.

By obvious observation, we know that if time t is large enough, then the last second term on the right-hand side of (A.17) will become arbitrarily small. Thus, in order to proceed further, we need to find a sufficiently large time $t\,(> T_\sigma)$, such that the following inequality of t holds

$$\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}\bar{V} \leqslant \frac{\lambda^2}{8c_3r(t)}.$$

This can be implied by (since (A.22) and (A.23))

$$\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}\bar{V} \leqslant \frac{\lambda^2}{8c_3c_5(\frac{5c_6}{4}(t - T_\sigma) + L^{\frac{5}{4}}(T_\sigma))},$$

or equivalently

$$8c_3c_5\bar{V}\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}(\frac{5c_6}{4}(t - T_\sigma) + L^{\frac{5}{4}}(T_\sigma)) \leqslant \lambda^2,$$

which is further implied by (since $\mathrm{e}^{\mathrm{T}}au \geqslant \tau$, $\forall\tau \in \mathbb{R}^+$)

$$8c_3c_5\bar{V}\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}(\frac{5\sigma c_6}{4}\mathrm{e}^{\frac{1}{\sigma}(t - T_\sigma)} + L^{\frac{5}{4}}(T_\sigma)) \leqslant \lambda^2.$$

Therefore, if $\sigma \in (0, 1)$ is chosen sufficiently small such that

$$10\sigma c_3c_5c_6\bar{V} < \lambda^2 \leqslant 8c_3c_5\bar{V}L^{\frac{5}{4}}(T_\sigma) + 10\sigma c_3c_5c_6\bar{V}, \quad \text{(A.24)}$$

for any prescribed λ, then by solving the above inequality of t, we obtain

$$t \geqslant \widehat{T}_\sigma =: T_\sigma + \sigma\ln\frac{8c_3c_5\bar{V}L^{\frac{5}{4}}(T_\sigma)}{\lambda^2 - 10\sigma c_3c_5c_6\bar{V}}, \quad \text{(A.25)}$$

which evidently ensures that $\hat{T}_\sigma \geqslant T_\sigma$ and

$$\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}\bar{V} \leqslant \frac{\lambda^2}{8c_3r(t)}, \ \forall t \geqslant \widehat{T}_\sigma.$$

Moreover, by (A.23)

$$\begin{aligned}& L(\widehat{T}_\sigma) \leqslant \\ & (\frac{5c_6}{4}\sigma\ln\frac{8c_3c_5\bar{V}L^{\frac{5}{4}}(T_\sigma)}{\lambda^2 - 10\sigma c_3c_5c_6\bar{V}} + L^{\frac{5}{4}}(T_\sigma))^{\frac{4}{5}} \leqslant \\ & (\frac{5c_6}{4}\sigma\ln\frac{8c_3c_5\bar{V}L^{\frac{5}{4}}(T_\sigma)}{\lambda^2 - 10\sigma c_3c_5c_6\bar{V}})^{\frac{4}{5}} + L(T_\sigma).\end{aligned} \quad \text{(A.26)}$$

Noting that (A.15), we have $\lim\limits_{\sigma\to 0^+}\dfrac{5c_6}{4}\sigma\ln\dfrac{8c_3c_5\bar{V}L^{\frac{5}{4}}(T_\sigma)}{\lambda^2 - 10\sigma c_3c_5c_6\bar{V}} = 0$. Then by $L(\widehat{T}_\sigma) \geqslant L(T_\sigma) \geqslant 1$ and (A.26), we know that when σ sufficiently small (see Remark A.1 below), there holds

$$L(T_\sigma) \leqslant L(\widehat{T}_\sigma) \leqslant (\frac{4}{3})^{\frac{4}{5}}L(T_\sigma), \quad \text{(A.27)}$$

and in this situation,

$$\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}\bar{V} \leqslant \frac{\lambda^2}{8c_3c_5L^{\frac{5}{4}}(T_\sigma)} \leqslant \frac{\lambda^2}{6c_3c_5L^{\frac{5}{4}}(\widehat{T}_\sigma)}, \ \forall t \geqslant \widehat{T}_\sigma.$$

For the first term on the right-hand side of (A.17), by (A.20) and (A.27), when $\sigma \in (0, 1)$ small enough, we know $\sigma^{\frac{1}{4}}L^{\frac{1}{4}}(\widehat{T}_\sigma) \leqslant (\frac{4}{3})^{\frac{1}{5}}\sigma^{\frac{1}{4}}L^{\frac{1}{4}}(T_\sigma) \leqslant (\frac{4}{3})^{\frac{1}{5}}(1 + 2\Theta + 2/c_2)^{\frac{1}{4}}$, and therefore,

$$\begin{aligned}\frac{\sigma\Theta}{L(T_\sigma)} \leqslant & (\frac{4}{3})^{\frac{4}{5}}\frac{\sigma^{\frac{3}{4}}\Theta(\sigma^{\frac{1}{4}}L^{\frac{1}{4}}(\widehat{T}_\sigma))}{L^{\frac{5}{4}}(\widehat{T}_\sigma)} \leqslant \\ & \frac{\sigma^{\frac{3}{4}}}{L^{\frac{5}{4}}(\widehat{T}_\sigma)} \cdot \frac{4(1 + 2\Theta + 2/c_2)^{\frac{1}{4}}\Theta}{3} \leqslant \\ & \frac{\lambda^2}{6c_3c_5L^{\frac{5}{4}}(\widehat{T}_\sigma)}.\end{aligned} \quad \text{(A.28)}$$

In virtue of above estimations, for any prescribed tracking level $\lambda > 0$, one can always choose sufficient small $\sigma > 0$, such that for any $t \geqslant \widehat{T}_\sigma$,

$$\begin{cases}\dfrac{\sigma\Theta}{L(T_\sigma)} \leqslant \dfrac{\lambda^2}{6c_3c_5L^{\frac{5}{4}}(\widehat{T}_\sigma)}, \\ \mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}\bar{V} \leqslant \dfrac{\lambda^2}{6c_3c_5L^{\frac{5}{4}}(\widehat{T}_\sigma)},\end{cases}$$

from which together with (A.22) and the definitions of V and c_3, it is concluded that for $\forall t \geqslant \widehat{T}_\sigma$,

$$\begin{aligned}& 2\varepsilon_1^2(t) + 2\|\eta(t)\|^2 \leqslant c_3V(t) \leqslant \\ & \frac{\lambda^2}{3c_5L^{\frac{5}{4}}(\widehat{T}_\sigma)} < \frac{\lambda^2}{2c_5L^{\frac{5}{4}}(\widehat{T}_\sigma)} \leqslant \frac{\lambda^2}{2r(\widehat{T}_\sigma)}.\end{aligned} \quad \text{(A.29)}$$

This is the extremely important inequality in the entire proof, based on which we can establish a contradiction to the foregoing supposed $L(+\infty) = +\infty$ and finally prove the boundedness of L, as will be proceeded in the later.

Due to the continuity of the closed-loop solution on $[0, +\infty)$ (already discussed) and $L(+\infty) = +\infty$ (supposed above), there always exists a finite time $T' > \widehat{T}_\sigma$, such that $L(T') > L(\widehat{T}_\sigma)$. Then by (A.29) and the updating law of L, there must exist times on $(\widehat{T}_\sigma, T')$, at which $2\varepsilon_1^2 2\varepsilon_1^2 + 2\|\eta\|^2 - \dfrac{\lambda^2}{2c_5 L^{\frac{5}{4}}} = 0$. In these times, we use T'' denote the first time (or the minimal time), at and before which, there hold respectively

$$\begin{cases} 2\varepsilon_1^2(T'') + 2\|\eta(T'')\|^2 = \dfrac{\lambda^2}{2c_5 L^{\frac{5}{4}}(T'')}, \\ 2\varepsilon_1^2(t) + 2\|\eta(t)\|^2 < \dfrac{\lambda^2}{2c_5 L^{\frac{5}{4}}(t)},\ \forall t \in [\widehat{T}_\sigma,\, T''). \end{cases} \tag{A.30}$$

The first relationship concludes $L(T'') > L(\widehat{T}_\sigma)$. Since otherwise, we know $L(T'') = L(\widehat{T}_\sigma)$ and in turn $2\varepsilon_1^2(T'') + 2\|\eta(T'')\|^2 < \dfrac{\lambda^2}{2c_5 L^{\frac{5}{4}}(\widehat{T}_\sigma)} = \dfrac{\lambda^2}{2c_5 L^{\frac{5}{4}}(T'')}$ by (A.29), which contradicts the first inequality of (A.30). On the other hand, from (A.22) and the second inequality of (A.30), we have for $\forall\, t \in [\widehat{T}_\sigma,\, T'')$,

$$2\varepsilon_1^2(t) + 2\|\eta(t)\|^2 < \frac{\lambda^2}{2c_5 L^{\frac{5}{4}}(t)} = \frac{\lambda^2}{2L(t)c_5 L^{\frac{1}{4}}(t)} \leqslant \frac{\lambda^2}{2r(t)},$$

which together with the updating law of L leads to $\dot{L}(t) = 0$, $\forall t \in [\widehat{T}_\sigma,\, T'')$, and then by the continuity of the closed-loop system, we have $L(T'') = L(\widehat{T}_\sigma)$. This contradicts the just deduced $L(T'') > L(\widehat{T}_\sigma)$ from the first inequality and finally shows that $L(t)$ is bounded on $[0,\, +\infty)$.

The boundedness of ε, η and M can be immediately established by that of L and Proposition 3. From this and the defined transformation (9), we can obtain that x and $\hat{x}$ are bounded on $[0,\, +\infty)$, so is u by its expression (i.e., (8)). The proof of the lemma is complete.

Remark A.1 In above proof, constant $\sigma > 0$ is required to be sufficiently small several times to satisfy different conditions. Collecting all situations (see (A.24)−(A.26), (A.28)), we have

$$\begin{cases} 10\sigma c_3 c_5 c_6 \bar{V} < \lambda^2 \leqslant 8c_3 c_5 \bar{V} L^{\frac{5}{4}}(T_\sigma) + 10\sigma c_3 c_5 c_6 \bar{V}, \\ (\dfrac{4}{3})^{\frac{4}{5}} - 1 \geqslant (\dfrac{5c_6}{4}\sigma \ln \dfrac{8c_3 c_5 \bar{V} L^{\frac{5}{4}}(T_\sigma)}{\lambda^2 - 10\sigma c_3 c_5 c_6 \bar{V}})^{\frac{4}{5}}, \\ \lambda^2 \geqslant 8c_3 c_5 (1 + 2\Theta + 1/c_2)^{\frac{1}{4}} \Theta \sigma^{\frac{3}{4}}. \end{cases}$$

Noting $L(T_\sigma) = 1 + 2\Theta + \dfrac{2}{\sigma c_2}$, it is obvious that there always exists solution in $(0,\, 1)$ for the always exists solution in $(0,\, 1)$ for the above inequalities of σ.

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