第 31 卷第 7 期
2014 年 7 月

控 制 理 论 与 应 用
Control Theory & Applications

Vol. 31 No. 7
Jul. 2014

DOI: 10.7641/CTA.2014.40029

奇异线性系统在执行器饱和受限下不变集条件的改进

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摘要: 本文考虑饱和线性反馈下奇异线性系统扩大吸引域估计的问题. 根据每个输入是否饱和, 将输入空间分成若干子区域. 在每个子区域内部, 系统模型中没有显示的部分状态的时间导数可被显式表达. 利用含有全部系统状态的二次Lyapunov函数, 建立一组双线性矩阵不等式形式的改进的不变集条件. 该组条件下, 二次Lyapunov函数的水平集可诱导出一个吸引域估计. 为得到最大的吸引域估计, 构建了以这些双线性矩阵不等式为约束条件的优化问题, 并为其求解给出了迭代算法. 仿真结果表明本文得到的吸引域估计明显大于现有结果.

关键词: 奇异系统; 执行器饱和; 稳定性分析; 吸引域; 不变集

中图分类号: TP273　　　**文献标识码**: A

Improved set invariance conditions for singular linear systems subject to actuator saturation

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Abstract: This paper considers the problem of enlarging the estimate for the domain of attraction of singular linear systems with saturated linear feedback. We partition the input space into several regions. In the interior of each of these regions, the time derivatives of partial states, which are not present in the system model, can be explicitly expressed. A quadratic Lyapunov function of all states of the system is employed to establish a set of conditions under which a level set of this quadratic Lyapunov function is contractively invariant with respect to the singular system, and thus results in an estimate of the domain of attraction. These conditions can be expressed in terms of bilinear matrix inequalities (BMIs). Based on these BMIs, a constrained optimization problem is formulated for obtaining the largest such estimate of the domains of attraction. An iterative algorithm is developed to solve this BMI problem. Simulation results show that the estimate thus obtained is significantly larger than an existing estimate.

Key words: singular systems; actuator saturation; stability analysis; domain of attraction; set invariance

1 Introduction

Over the past decades, considerable attention has been devoted to the study of singular linear systems, also called descriptor systems. Singular linear systems, described by a set of differential and algebraic equations, are used in modeling many practical systems, such as power systems, electrical networks, biological systems, social economic systems[1–2]. A large number of results on fundamental concepts[1–2], stability and stabilization [3–6], as well as performance of such systems[7–9] have been reported in the literature. Many control problems for non-singular systems have also been formulated and effectively solved as problems for singular systems[10–11].

Actuator saturation is ubiquitous in practical control systems. The presence of actuator saturation can lead to the performance degradation or, in the extreme case, instability, of control systems. In recent years, much attention from research community has been paid to the analysis and synthesis of control systems with actuator saturation. Global asymptotic stabilization and semi-global asymptotic stabilization of open loop systems that are not exponentially unstable have been investigated in [12–15], and local stability and stabilization have been studied extensively for exponentially unstable open loop systems (see [16–22] for a small sample of the literature). The majority of the literature on control systems with actuator saturation focus on non-singular systems.

Deriving from extensive studies on both singular lin-

Received 14 January 2014; accepted 19 April 2014.
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This work was supported in part by the National Natural Science Foundation of China (Nos. 61221003, 61273105).

ear systems and non-singular linear systems subject to actuator saturation, several results have been obtained in recent years on the stability analysis and stabilization of singular linear, or even nonlinear, systems subject to actuator saturation[23–26]. Specially, the problem of estimating the domain of attraction of singular systems with actuator saturation has been drawn much interest[4, 27–31]. For example, sufficient conditions were established in [27] under which the closed-loop system under a given saturated linear partial state feedback is locally asymptotically stable. Conditions in the form of linear matrix inequalities (LMIs) were established in [4] under which an ellipsoid is contractively invariant, and thus can be used as an estimate of the domain of attraction, for a singular linear systems with input saturation. The set invariance problem for Lipschitz nonlinear singular systems with actuator saturation has been considered in [3], and sufficient conditions have been established to guarantee the contractive invariance of the closed-loop system with respect to a given ellipsoid. Further, $\mathcal{L}_2$ gain and $\mathcal{L}_\infty$ performance analysis and design for singular linear systems with actuator saturation were carried out in [32].

In this paper, we consider the problem of estimating the domain of attraction for a singular linear system subject to actuator saturation. In the existing literature where the same problem has been addressed[4, 27–28, 32], a quadratic Lyapunov function of the partial state whose time-derivative is present in the system model is adopted to obtain an estimate of the domain of attraction in the form of a contractively invariant level set of the quadratic function, which is an ellipsoid. These Lyapunov functions do not involve the remaining part of the state, whose time derivative is not present in the system model. Less conservative results could be expected if the partial state whose time derivative is not present in the system model is also reflected in the Lyapunov function. To explore this possibility for improvement, we will unearth further details inherent in the differential and algebraic equations of the system model. To this end, we will partition the input space into several regions according to the saturation status of each input. In the interior of each of these regions, the time derivative of each state whose time derive is not present in the system model can be explicitly expressed. Thus, an alternative system model can be formed where the derivatives of all states are present. A quadratic Lyapunov function of all system states will then be adopted whose time derivative can be evaluated along the trajectory of the full state of the system. Clearly, this Lyapunov function provides addition degrees of freedom in estimating the domain of attraction. Conditions will be established under which a level set of the Lyapunov function is contractively invariant with respect to the singular system and is thus an estimate of the domain of attraction. These conditions improve the existing ones established in [4]. These conditions can be expressed in terms of bilinear matrix inequalities (BMIs). Based on these BMIs, a constrained optimization problem is formulated for arriving at the largest such estimate of the domain of attraction. An iterative algorithm is developed to solve this BMI problem. Simulation results show that the estimate thus obtained is significantly larger than the estimate obtained by using the method of [4].

The remainder of our paper is organized as follows. In Section 2, we recall some basic definitions of singular systems and the convex hull representation of a saturated linear feedback. We will derive an expression for the time derivative of the partial state whose derivative is not present in the system model. In Section 3, a quadractic Lyapunov function of the full state of the system is constructed and conditions are established under which a level set of the constructed Lyapunov function, which is an ellipsoid, is contractively invariant and thus results in an estimate of the domain of attraction. An optimization problem with BMI constraints is formulated to maximize this estimate of the domain of attraction. Section 4 provides some simulation results to illustrate the effectiveness of the results in Section 3. Section 5 concludes the paper.

Notation Denote I_m as the identity matrix of dimension m, and $0_{n\times m}$ as the $n\times m$ zero matrix. For a square matrix A, $\mathrm{He}(A) := A + A^{\mathrm{T}}$. For two integers l_1 and $l_2 \geqslant l_1$, $I[l_1, l_2]$ denotes the set of integers $\{l_1, l_1+1, \cdots, l_2\}$. For an integer m, let $\mathcal{D}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{D}$. Suppose that these elements of $\mathcal{D}$ are labeled as D_i, $i \in I[1, 2^m]$. Without loss of generality, we denote $D_1 = I_m$, and $D_{2^m} = 0_{m\times m}$. Let $D_i^- = I - D_i$. Clearly, $D_i^- \in \mathcal{D}$. Let d_{ij} be the jth diagonal element of D_i. Let $\mathcal{J}_i = \{j:\ d_{ij} \neq 0,\ j \in I[1, m]\}$ and $\mathcal{J}_i^- = I[1, m] \setminus \mathcal{J}_i$. Let J_i^- be the number of the elements of $\mathcal{J}_i^-$. For $i \neq 1$, we denote the jth column of D_i^- as c_{ij}, and let c_i be the matrix formed by all c_{ij} such that $j \in \mathcal{J}_i^-$. Clearly, $c_i \in \mathbb{R}^{m\times J_i^-}$. In addition, we denote $c_1 = 0_{m\times 1}$ and $J_1^- = 1$. Clearly, $c_i c_i^{\mathrm{T}} = D_i^-$. For a matrix $P \in \mathbb{R}^n$ with $P = P^{\mathrm{T}} > 0$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n:\ x^{\mathrm{T}} P x \leqslant 1\}$.

2 Preliminaries

Consider a singular linear system under a saturated linear state feedback

$$\begin{cases} E\dot{x} = Ax + B\mathrm{sat}(u), \\ u = Fx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $\mathrm{sat}: \mathbb{R}^m \to \mathbb{R}^m$ is the vector valued standard saturation function

$$\mathrm{sat}(u) = [\mathrm{sat}(u_1)\ \ \mathrm{sat}(u_2)\ \ \cdots\ \ \mathrm{sat}(u_m)]^{\mathrm{T}},$$

with $\mathrm{sat}(u_j) = \mathrm{sgn}\, u_j \min\{1, |u_j|\}, j \in I[1, m]$. A signal u_j is said to saturate if $|u_j| \geqslant 1$, and it is said to unsaturate of $|u_j| < 1$. u_j is said to critically saturate if $|u_j| = 1$. Let $\mathrm{rank}(E) = q$. It is also without loss generality to assume that (E, A, B, F) are in the following form,

$$E = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ F = [F_1\ \ F_2].$$

We will partition the state vector accordingly as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ x_1 \in \mathbb{R}^q,\ x_2 \in \mathbb{R}^{n-q}.$$

Before stating our basic assumptions on system (1), we recall some basic definitions for singular linear systems. The open loop system (E, A) is said to be regular if $\det(sE - A)$ is not identically zero. The system (E, A) is said to be impulse free if $\deg(\det(sE-A)) = \mathrm{rank}(E)$. The system (E, A) is said to be stable if $\{s \in \mathbb{C}, \det(sE-A) = 0\} \subset \{s \in \mathbb{C}, \mathrm{Re}(s) < 0\}$, where $\mathbb{C}$ is the set of all complex numbers. It has been established in [4] that system (1) is regular and impulse free if the matrix $A_{22} + B_2 D_i F_2$ is nonsingular for each $i \in I[1, 2^m]$. In this paper, we will assume that $A_{22} + B_2 D_i F_2$ is nonsingular and system $(E, A + BF)$, which describes system (1) in the absence of actuator saturation, is stable. The stability of $(E, A + BF)$ guarantees the existence of the domain of attraction of the origin for system (1).

We next recall the treatment of a saturated linear feedback $\mathrm{sat}(Fx)$ from [17]. The saturated linear feedback $\mathrm{sat}(Fx)$ can be expressed on a convex hull of a group of auxiliary linear feedbacks. For an $H \in \mathbb{R}^{m\times n}$, let

$$\mathcal{L}(H) = \{x \in \mathbb{R}^n \colon |h_j x| \leqslant 1, j \in I[1, m]\},$$

where h_j represents the jth row of H. We note that $\mathcal{L}(H)$ denotes the region in $\mathbb{R}^n$ where Hx does not saturate. The following lemma is adopted from [17].

Lemma 1 Let $F, H \in \mathbb{R}^{m\times n}$. Then, for any $x \in \mathcal{L}(H)$,

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_i Fx + D_i^- Hx, i \in I[1, 2^m]\right\},$$

where co stands for the convex hull.

This representation of the saturated linear feedback $\mathrm{sat}(Fx)$ has been extensively used to solve various control problems for linear systems under actuator saturation, such as stability analysis, set invariance analysis, and $\mathcal{L}_2$ gain analysis[4, 17–19, 32]. These control problems can be cast into and solved as optimization problems involving as part of the constraints 2^m linear or bilinear matrix inequalities, each of which is associated with a vertex of the convex hull in Lemma 1.

Note that each input u_j contains two different statuses, saturation and non-saturation. Thus we could partition the input space into 2^m regions according to the saturation status of each input u_j. We use $\mathcal{D}_i$, $i \in I[1, 2^m - 1]$, to denote the region in which, for any $j \in \mathcal{J}_i$, the jth input does not saturate. In addition, we denote the region where all inputs saturate as $\mathcal{D}_{2^m}$. Clearly, a region $\mathcal{D}_i$ is associated with a unique D_i. Let $\mathcal{D}_i^\circ$ be the interior of $\mathcal{D}_i$, $i \in I[1, 2^m]$.

Consider the algebraic constraint of the closed-loop system (1)

$$0 = A_{21}x_1 + A_{22}x_2 + B_2\mathrm{sat}(Fx). \tag{2}$$

If $Fx \in \mathcal{D}_i^\circ$, $i \in I[1, 2^m]$, the algebraic constraint (2) can be rewritten as

$$\begin{aligned} 0 = {} & A_{21}x_1 + A_{22}x_2 + B_2 D_i F_1 x_1 + \\ & B_2 D_i F_2 x_2 + B_2 c_i \mathrm{sat}(c_i^{\mathrm{T}} Fx). \end{aligned} \tag{3}$$

Note that each element of $\mathrm{sat}(c_i^{\mathrm{T}} Fx) \in \mathbb{R}^{J_i^-}$ is either 1 or -1 if $Fx \in \mathcal{D}_i^\circ$. Thus the time derivative of $\mathrm{sat}(c_i^{\mathrm{T}} Fx)$ exists when $Fx \in \mathcal{D}_i^\circ$, and

$$\frac{\mathrm{dsat}(c_i^{\mathrm{T}} Fx)}{\mathrm{d}t} = 0.$$

By differentiating both sides of (3) we obtain

$$0 = (A_{21} + B_2 D_i F_1)\dot{x}_1 + (A_{22} + B_2 D_i F_2)\dot{x}_2.$$

Denote $\mathcal{A}_{ikl} = A_{kl} + B_k D_i F_l$, $k, l = 1, 2$, $i \in I[1, 2^m]$. Since, by the assumption we made in Section 2, matrices $\mathcal{A}_{i22} = A_{22} + B_2 D_i F_2$ are nonsingular, we have

$$\begin{aligned} \dot{x}_2 = {} & -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21}\dot{x}_1 = \\ & -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21}(A_{11}x_1 + A_{12}x_2 + B_1\mathrm{sat}(Fx)) = \\ & -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21}((A_{11} + B_1 D_i F_1)x_1 + \\ & (A_{12} + B_1 D_i F_2)x_2 + B_1 c_i \mathrm{sat}(c_i^{\mathrm{T}} Fx)) = \\ & -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21}(\mathcal{A}_{i11}x_1 + \mathcal{A}_{i12}x_2 + B_1 c_i \mathrm{sat}(c_i^{\mathrm{T}} Fx)). \end{aligned}$$

Noting that the algebraic constraint (2) is equivalent to $0 = \mathcal{A}_{i21}x_1 + \mathcal{A}_{i22}x_2 + B_2 c_i \mathrm{sat}(c_i^{\mathrm{T}} Fx)$, we obtain the following system, which is an alternative form of system (1) and contains the explicit expression of $\dot{x}_2$,

$$\mathcal{E}\dot{x} = \mathcal{A}_i x + \mathcal{B}_i \mathrm{sat}(\mathcal{F}_i x), i \in I[1, 2^m], \tag{4}$$

where

$$\mathcal{E} = \begin{bmatrix} I_n \\ 0_{(n-q)\times n} \end{bmatrix},$$

$$\mathcal{A}_i = \begin{bmatrix} \mathcal{A}_{i11} & \mathcal{A}_{i12} \\ -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21}\mathcal{A}_{i11} & -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21}\mathcal{A}_{i12} \\ \mathcal{A}_{i21} & \mathcal{A}_{i22} \end{bmatrix},$$

$$\mathcal{B}_i = \begin{bmatrix} B_1 c_i \\ -\mathcal{A}_{i22}^{-1}\mathcal{A}_{i21} B_1 c_i \\ B_2 c_i \end{bmatrix}, \ \mathcal{F}_i = c_i^{\mathrm{T}} F.$$

For the singular linear system under actuator saturation (1), we are interested in the domain of attraction of its equilibrium at the origin, which is the set of all compatible initial conditions from which the trajectories converge to the origin. In this paper, we will use the following form of estimate of the domain of attraction

$$\begin{aligned} \mathcal{E}_V(x) := \{ & x \in \mathbb{R}^n : x \in \mathcal{E}(P), \\ & A_{21}x_1 + A_{22}x_2 + B_2\mathrm{sat}(Fx) = 0\}, \end{aligned}$$

where $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix. This estimate involves a quadratic Lyapunov function of the form

$$V(x) = x^{\mathrm{T}} Px.$$

Let $\dot{V}(x)$ denote the time derivative of $V(x)$ whenever it exists. Since $\dfrac{\mathrm{dsat}(c_i^{\mathrm{T}} Fx)}{\mathrm{d}t}$ exists if no u_j saturates critically, we conclude from (3) that $\dot{x}_2$ exists for almost all $x_1 \in \mathbb{R}^q$, and thus $\dot{V}(x)$ exists for almost all $x \in \mathbb{R}^n$. If $\dot{V}(x) < 0$ for almost all x in a set $\mathcal{S}$ containing the origin, then $\mathcal{S}$ is contractively invariant, that is, the origin is locally asymptotically stable for system (1) and all trajectories starting compatible initial conditions in $\mathcal{S}$ will tend to the origin[16].

3 Main results

In this section, we will utilize the expression for $\dot{x}_2$ obtained in Section 2 to establish conditions under which $\mathcal{E}_V(P)$ is a contractively invariant set of system (1).

3.1 Set Invariance conditions

To present the main result of this section, we construct the following matrices

$$\mathcal{P}_i = [P\ \ \bar{P}_i],\ \bar{P}_i = \begin{bmatrix} P_{1i} \\ P_{2i} \end{bmatrix},\ i \in I[1, 2^m], \tag{5}$$

where $P \in \mathbb{R}^{n\times n}$ is positive definite, $P_{1i} \in \mathbb{R}^{q\times(n-q)}$ and $P_{2i} \in \mathbb{R}^{(n-q)\times(n-q)}$. The following theorem establishes set invariance conditions for $\mathcal{E}_V(P)$.

Theorem 1 For a given $P > 0$, if there exist matrices $\mathcal{P}_i$ of the form (5) and matrices $\mathcal{H}_i \in \mathbb{R}^{J_i^- \times n}$, $i \in I[1, 2^m]$, such that

$$\begin{aligned} &\mathrm{He}\left(\mathcal{P}_i(\mathcal{A}_i + \mathcal{B}_i(D_{k_i}\mathcal{F}_i + D_{k_i}^-\mathcal{H}_i))\right) < 0, \\ &k_i \in I[1, 2^{J_i^-}], i \in I[1, 2^m], \end{aligned} \tag{6}$$

and $\mathcal{E}(P) \subseteq \mathcal{L}(\mathcal{H}_i)$, $i \in I[1, 2^m]$, then $\dot{V}(x) < 0$ for almost all $x \in \mathcal{E}_V(P)$, and thus, the set $\mathcal{E}_V(P)$ is a contractively invariant set of system (1).

Proof Before we set off to prove the theorem, we observe that, by the assumption that $A_{22}+B_2D_iF_2$ is nonsingular for all $i \in I[1, 2^m]$, system (1) is regular and impulse free.

To prove the theorem, we will show that that $\dot{V}(x) < 0$ for all $x \in \mathcal{E}_V(P) \setminus \{0\}$ such that $Fx \in \mathcal{D}_i^\circ$, $i \in I[1, 2^m]$. By the definition of $\mathcal{E}_V(P)$, it is clear that $\mathcal{E}_V(P) \subset \mathcal{E}(P)$. Then, for all $i \in I[1, 2^m]$, condition $\mathcal{E}(P) \subseteq \mathcal{L}(\mathcal{H}_i)$ implies that $\mathcal{E}_V(P) \subset \mathcal{L}(\mathcal{H}_i)$. By Lemma 1, for every $x \in \mathcal{E}_V(P)$ such that $Fx \in \mathcal{D}_i^\circ$,

$$\mathrm{sat}(\mathcal{F}_i x) \in \mathrm{co}\{D_{k_i}\mathcal{F}_i x + D_{k_i}^-\mathcal{H}_i x : k_i \in I[1, 2^{J_i^-}]\}.$$

where D_{k_i}'s are the $J_i^- \times J_i^-$ diagonal matrices whose diagonal elements are either 1 or 0. It follows that

$$\begin{aligned} &\mathcal{A}_i x + \mathcal{B}_i \mathrm{sat}(\mathcal{F}_i x) \in \\ &\mathrm{co}\{(\mathcal{A}_i + \mathcal{B}_i D_{k_i}\mathcal{F}_i x + \mathcal{B}_i D_{k_i}^-\mathcal{H}_i)x : k_i \in I[1, 2^{J_i^-}]\}. \end{aligned} \tag{7}$$

Since $\dot{x}_2$ exists on every $\mathcal{D}_i^\circ$, the time derivative of the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ along the trajectory of the closed loop system (1) is given by

$$\begin{aligned} \dot{V}(x) = &\dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = \\ &\dot{x}^{\mathrm{T}}\mathcal{E}^{\mathrm{T}}\mathcal{P}_i^{\mathrm{T}}x + x^{\mathrm{T}}\mathcal{P}_i\mathcal{E}\dot{x} = \\ &2x^{\mathrm{T}}\mathcal{P}_i(\mathcal{A}_i + \mathcal{B}_i \mathrm{sat}(\mathcal{F}_i x)). \end{aligned} \tag{8}$$

By (7), we have

$$\begin{aligned} &\dot{V}(x) = 2x^{\mathrm{T}}\mathcal{P}_i(\mathcal{A}_i + \mathcal{B}_i \mathrm{sat}(\mathcal{F}_i x)) \leqslant \\ &\max_{k_i \in I[1, 2^{j_i^-}]} 2x^{\mathrm{T}}\mathcal{P}_i(\mathcal{A}_i + \mathcal{B}_i D_{k_i}\mathcal{F}_i x + \mathcal{B}_i D_{k_i}^-\mathcal{H}_i)x, \end{aligned}$$

for any $x \in \mathcal{E}_V(P)$ such that $u \in \mathcal{D}_i^\circ$. In view of (6), we have $\dot{V}(x) < 0$ for all $x \in \mathcal{E}_V(P)\backslash\{0\}$ such that $Fx \in \mathcal{D}_i^\circ$. It follows that $\dot{V}(x) < 0$ for almost all $x \in \mathcal{E}_V(P)$, which indicates that the set $\mathcal{E}_V(P)$ is contractively invariant for system (1).

Theorem 1 presents a set of sufficient conditions under which $\mathcal{E}_V(P)$ is a contractively invariant set for system (1). If we set

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix},\ \bar{P}_i = \begin{bmatrix} P_3 \\ P_4 \end{bmatrix},$$

where $P_1 \in \mathbb{R}^{q\times q}$ is positive definite, $P_3 \in \mathbb{R}^{n\times(n-q)}$ and $P_4 \in \mathbb{R}^{(n-q)\times(n-q)}$, the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ will degenerate to $V_L(x) = x_1^{\mathrm{T}}P_1x_1$. Moreover, the presence of $\dot{x}_2$ in $x^{\mathrm{T}}\mathcal{P}_i\mathcal{E}\dot{x}$ in (8) will vanish and the term $x^{\mathrm{T}}\mathcal{P}_i\mathcal{E}\dot{x}$ becomes $x^{\mathrm{T}}\bar{P}E\dot{x}$, where

$$\bar{P} = \begin{bmatrix} P_1 & P_3 \\ 0 & P_4 \end{bmatrix}.$$

This results in the conditions of Theorem 1 in [4]. In other words, Theorem 1 in [4] is a special case of Theorem 1 of the current paper, The generalization is made possible by two factors. On the one hand, the quadratic Lyapunov function adopted in the current paper can be viewed as a generalization of $V_L(x) = x_1^{\mathrm{T}}P_1x_1$ used in [4]. On the other hand, we explore the information of $\dot{x}_2$ in the interior of every region in the input space, and the generalized quadratic Lyapunov function effectively incorporates this information. Both of these two factors result from the exploration of $\dot{x}_2$ and lead to the less conservativeness of conditions in Theorem 1 than those in Theorem 1 in [4].

3.2 Estimation of the domain of attraction

The set $\mathcal{E}_V(P)$ satisfying conditions in Theorem 1 is a contractively invariant set of system (1), and thus can be used as an estimate of the domain of attraction for it. Obtaining a maximized estimate of the domain of attraction then boils down to the determination of the largest contractively invariant set $\mathcal{E}_V(P)$. Recall the definition of $\mathcal{E}_V(P)$. It is clear that $\mathcal{E}_V(P)$ is not a convex set due to the algebraic constraint (2). The size of $\mathcal{E}_V(P)$ can be measured with respect to a shape reference set $\mathcal{R}$ by the largest α such that $\alpha\mathcal{R} \subseteq \mathcal{E}(P_{ai_0})$ for some $i_0 \in I[1, 2^m]$, where $P_{ai_0} = M_{i_0}^{\mathrm{T}}PM_{i_0}$ and

$$M_{i_0} = \begin{bmatrix} I_q \\ -\mathcal{A}_{i_0 22}^{-1}\mathcal{A}_{i_0 21} \end{bmatrix}.$$

Although the relationship between $\mathcal{E}(P_{ai_0})$ and $\mathcal{E}_V(P)$ is not clear, our simulation experience shows the effectiveness of this measurement. Let $\mathcal{R}$ be a polyhedron of the form $\mathcal{R} = \{r_1, r_2, \cdots, r_p\}$, $r_l \in \mathbb{R}^n$, $l \in I[1, p]$. Then $\alpha\mathcal{R} \subseteq \mathcal{E}(P_{ai_0})$ is equivalent to $r_l^{\mathrm{T}}M_{i_0}^{\mathrm{T}}PM_{i_0}r_l \leqslant \gamma$, $l \in I[1, q]$, where $\gamma = \dfrac{1}{\alpha^2}$. In what follows, we formulate an optimization problem based on Theorem 1 for a maximized estimate of the domain of attraction of system (1):

$$\min_{P>0, P_{1i}, P_{2i}, \mathcal{H}_i, i\in I[1,2^m], i_0\in I[1,2^m]} \gamma, \tag{9}$$

$$\begin{aligned} \text{s.t. a) } &r_l^{\mathrm{T}}M_{i_0}^{\mathrm{T}}PM_{i_0}r_l \leqslant \gamma, l \in I[1, p], \\ \text{b) } &\mathrm{He}(\mathcal{P}_i(\mathcal{A}_i + \mathcal{B}_i D_{k_i}\mathcal{F}_i + \mathcal{B}_i D_{k_i}^-\mathcal{H}_i)) < 0, \\ &k_i \in I[1, 2^{J_i^-}], i \in I[1, 2^m], \end{aligned}$$

$$\text{c)}\ \begin{bmatrix} 1 & h_{ij} \\ h_{ij}^{\mathrm{T}} & P \end{bmatrix} \geqslant 0,\ j \in I[1, J_i^-],\ i \in I[1, 2^m],$$

where Constraint c) is equivalent to $\mathcal{E}(P) \subseteq \mathcal{L}(\mathcal{H}_i),\ i \in I[1, 2^m]$, and h_{ij} is the jth row of $\mathcal{H}_i$.

Since the inequalities in Constraint b) contain product terms among pairs of the unknown matrices, the optimization problem (9) is a bilinear matrix inequality (BMI) problem, whose global solution is hard to obtain. Various iterative algorithms have been developed to deal with BMI problems, such as the direct iteration method and the path-following method. In this paper, we will use the direct iteration method to solve the optimization problem (9). The resulting iterative algorithm is given as follows.

Algorithm 1 Solution of optimization problem (9).

Step 1 Solve the LMI optimization problem derived in [4],

$$\min_{Q_1>0, G_1, Q_3, Q_4} \gamma, \tag{10}$$

$$\text{s.t. a)}\ \begin{bmatrix} \gamma & r_l^{\mathrm{T}} \\ r_l & Q_1 \end{bmatrix} \geqslant 0,\ l \in I[1, p],$$

$$\text{b)}\ \mathrm{He}((A + B(D_i F Q + D_i^- G))) < 0,\ i \in I[1, 2^m],$$

$$\text{c)}\ \begin{bmatrix} 1 & g_{1j} \\ g_{1j}^{\mathrm{T}} & Q_1 \end{bmatrix} \geqslant 0, j \in I[1, m],$$

where $G = [G_1\ \ 0_{m\times(n-q)}]$ and g_{1j} is the jth row of G_1. Denote the solution as (Q_1, G_1, Q_3, Q_4). Let $H = GQ_1^{-1}$. Set $\mathcal{H}_i = c_i^{\mathrm{T}} H$ for all $i \in I[1, 2^m]$. Let $s = 1$, and $S(s) = 0$. Set a small positive scalar δ.

Step 2 Set $s = s + 1$. Solve the following LMI optimization problem, which results from the optimization problem (9) with fixed $\mathcal{H}_i$'s,

$$\min_{P>0, P_{1i}, P_{2i}, i\in I[1,2^m], i_0 \in I[1,2^m]} \gamma, \tag{11}$$

$$\text{s.t. a)}\ r_l^{\mathrm{T}} M_{i_0}^{\mathrm{T}} P M_{i_0} r_l \leqslant \gamma,\ l \in I[1, p],$$

$$\text{b)}\ \mathrm{He}(\mathcal{P}_i(\mathcal{A}_i + \mathcal{B}_i D_{k_i} \mathcal{F}_i + \mathcal{B}_i D_{k_i}^- \mathcal{H}_i)) < 0,$$

$$k_i \in I[1, 2^{J_i^-}], i \in I[1, 2^m],$$

$$\text{c)}\ \begin{bmatrix} 1 & h_{ij} \\ h_{ij}^{\mathrm{T}} & P \end{bmatrix} \geqslant 0,\ j \in I[1, J_i^-],\ i \in I[1, 2^m].$$

Denote the solution as $(\gamma_{\mathrm{opt}}, P, P_{1i}, P_{2i})$. Let $S(s) = \gamma_{\mathrm{opt}}$. If $|S(s) - S(s-1)| \leqslant \delta$, stop, else, go to Step 3.

Step 3 Solve the following LMI optimization problem which is the optimization problem (9) with fixed P, P_{3i} and P_{4i},

$$\min_{\mathcal{H}_i, i\in[1,2^m]} \gamma, \tag{12}$$

$$\text{s.t. a)}\ \begin{bmatrix} 1 & h_{ij} \\ h_{ij}^{\mathrm{T}} & \gamma P \end{bmatrix} \geqslant 0,\ j \in I[1, J_i^-],\ i \in I[1, 2^m],$$

b) Constraints b) and c) in (9).

Denote the solution as $\mathcal{H}_i$. Go to Step 2.

In the above algorithm, the optimal solution of (10) from [4] is set as the initial values of the iteration procedure. Such a choice of the initial values does not guarantee that the solution obtained from our algorithm is globally optimal. However, since the initial values of the iteration procedure are inherited from the optimal solution derived with the existing set invariance conditions, the result obtained from our algorithm will be at least as good as that obtained from the existing set invariance conditions[4].

Compared with the optimization problem (10) with 2^m LMIs in Constraint b), the optimization problem (9) formulated in this paper contains $\sum\limits_{l=0}^{m} C_m^l 2^{m-l}$ BMIs in its Constraint b). Larger estimates can be obtained from (9) at the cost of heavier computational burden. Thus, a trade-off should be considered between the conservativeness of the results and the computational burden. If the number of saturated inputs is small, we can use Theorem 1 to obtain a larger estimate of the domain of attraction. However, if the number of saturated inputs is large, the approach in [4] can be adopted to avoid excessive computation.

4 A numerical example

In this section, a numerical example is provided to demonstrate the effectiveness of our proposed approach. Let us consider system (1) with the following parameters,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.6 & -0.8 & 0.5 \\ 0.8 & 0.6 & 4 \\ 0.6 & 1 & 0.8 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 0.5 & 2 \end{bmatrix}, F = \begin{bmatrix} -1 & -2 & -1 \\ 2 & -3 & -1 \end{bmatrix}.$$

To apply Algorithm 1, let $\mathcal{R} = \{r_1\},\ r_1 = [1\ \ 15]^{\mathrm{T}}$, and solve the LMI optimization problem (10) to obtain

$$P_1^{\text{Lin and Lv}} = \begin{bmatrix} 0.1194 & -0.0327 \\ -0.0327 & 0.0184 \end{bmatrix}.$$

We select

$$M_1 = \begin{bmatrix} I_q \\ \mathcal{A}_{122}^{-1}\mathcal{A}_{121} \end{bmatrix}$$

for the measurement of $\mathcal{E}_V(P)$. We carry out Algorithm 1 and obtain

$$P^{\text{Theorem1}} = \begin{bmatrix} 0.1145 & -0.0565 & -0.0130 \\ -0.0565 & 0.0748 & 0.0249 \\ -0.0130 & 0.0249 & 0.0087 \end{bmatrix}.$$

We plot both $\mathcal{E}(P_1^{\text{Lin and Lv}})$ and $\mathcal{E}_V(P^{\text{Theorem1}})$ in Fig.1 for comparison. As is apparent in this figure, the estimate based on Theorem 1, $\mathcal{E}_V(P^{\text{Theorem1}})$, is significantly larger than $\mathcal{E}(P_1^{\text{Lin and Lv}})$ resulting from Theorem 1 in [4]. This illustrates that the set invariance conditions in Theorem 1, where $\dot{x}_2$ was explored and used to establish these improved conditions, are less conservative than those in [4] without employing the explicit expression of $\dot{x}_2$. To verify the contractive invariance of $\mathcal{E}_V(P^{\text{Theorem1}})$, we plot in Fig.1 a converging trajectory starting from the boundary of $\mathcal{E}_V(P^{\text{Theorem1}})$. The evolutions of its states, the underling control inputs and the quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P^{\text{Theorem1}} x$ are respectively depicted in Figs.2−4.

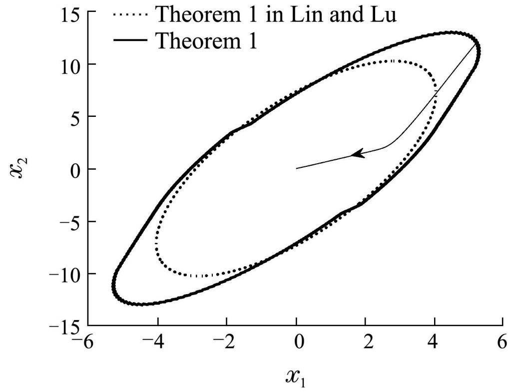

Fig. 1 The estimates of the domain of attraction and a converging trajectory starting from the boundary of $\mathcal{E}_V(P^{\text{Theorem1}})$.

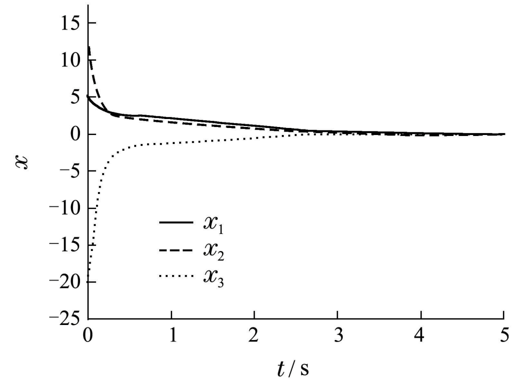

Fig. 2 The evolutions of the system states.

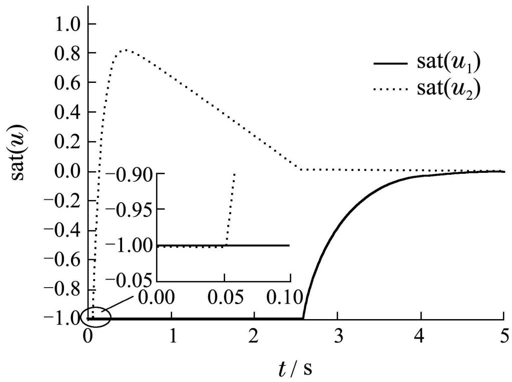

Fig. 3 The evolutions of the control inputs

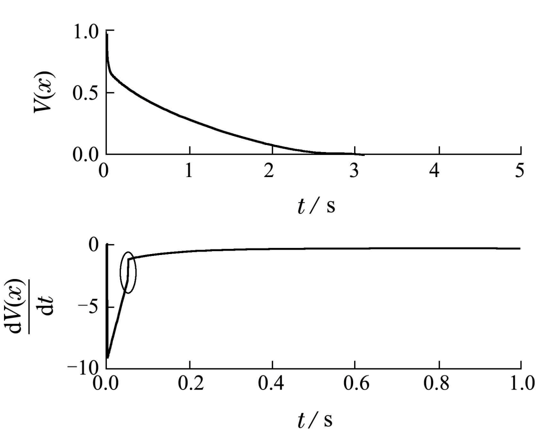

Fig. 4 The evolutions of the quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P^{\text{Theorem1}} x$ and its time derivative

Moreover, in the lower subplot of Fig.4, we can clearly see that a jump of signal at about $t = 0.05$s in the evolution of $\dot{V}(x)$. This implies that $\dot{V}(x)$ does not exist at that point in time. As we know, this phenomenon occurs when one of inputs critically saturates. A close observation of Fig.3 which shows that the input u_2 indeed critically saturates at about $t = 0.05$s manifests this point.

5 Conclusions

This paper revisited the problem of estimating the domain of attraction for a singular linear system subject to actuator saturation and proposed a new approach to solving the problem. We divided the input space into 2^m regions, and explored the information of the time derivative of x_2 in the interior of each of these regions. With the obtained explicit expression for the time derivative of x_2, a quadratic Lyapunov function of the full system state was utilized to establish conditions under which a level set of the Lyapunov function results in a contractively invariant set of the singular linear system with actuator saturation. These conditions cover the existing conditions in [4] as a special case and result in a significantly larger estimate of the domain of attraction than those[4] could. Simulation results demonstrate the effectiveness of our proposed approach.

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