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一类非齐次高阶非线性系统的自适应控制设计

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摘要: 本文讨论了一类具有未知参数的非齐次高阶非线性系统的全局强稳定自适应控制器的设计问题, 定义了一个时变未知参数和一列非齐次辅助函数, 通过增加辅助函数的积分项的方法并结合自适应控制技术, 放宽了目前已有结论中对非线性项的限制, 给出了使系统全局强稳定的连续自适应控制器存在的一个充分条件, 给出了它的自适应状态反馈控制器的递推设计方法, 并通过一个例子验证了文章的理论结果.

关键词: 非线性系统; 自适应系统; 非齐次; 未知控制系数

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Adaptive control design for a class of high order nonlinear nonhomogeneous uncertain systems

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Abstract: This paper discusses the problem of the globally strongly stablilizing adaptive controller design for a class of nonhomogeneous high order nonlinear systems with unknow parameters, by defining an unknown parameter which need dynamic updating and a series of nonhomogeneous auxiliary functions, also by using the method of adding an integrator of the nonhomogeneous auxiliary function merging with adaptive technique, the restriction on the nonlinearities of existing results is relaxed and the sufficient conditions for the existence of continuous adaptive controller which guarantees the systems are globally strongly stable is given. A recursive design procedure is provided to achieve continuous adaptive state feedback control law. Finally, a simulation example is provided to illustrate the correctness of the theoretical results.

Key words: nonlinear systems; adaptive systems; nonhomogeneous; unknown control coefficients

1 引言(Introduction)

本文研究形如

$$\begin{cases} \dot{x}_i = d_i(t,x,u,\theta)x_{i+1}^{p_i} + g_i(t,x,u,\theta), \\ \dot{x}_n = d_n(t,x,u,\theta)u^{p_n} + g_n(t,x,u,\theta) \end{cases} \tag{1}$$

的高阶非线性系统的自适应控制问题. 对这种形式的系统, Lin和Qian等学者改进了传统的反推方法, 在文献[1–2]中提出了增加幂次积分方法, 初步解决了该系统的控制问题. 利用该方法, 对非线性高阶系统的研究取得了一系列成果[3–7]. 文献[3]在控制系数$d_i(\cdot)$下界为已知常数或已知光滑函数且非线性项$g_i(\cdot)$ 满足一定增长条件的情况下, 利用Young不等式进行变量分离, 给出了系统连续状态反馈控制器的设计方法, 文献[4–5]则在系统含有未知参数的情况下, 分别给出了系统的连续和光滑的自适应控制器的设计方法. 在此基础上, Sun和Liu在文献[6–7]中讨论了控制系数下界未知的情况, 通过定义至少$n+1$个未知参数, 给出了其自适应控制器的设计方法. 为了避免文献[7]中过参数的问题, Zhang和Liu在文献[8]中结合调节函数的方法, 仅使用一个参数就给出了其自适应控制的设计方法. 此外, 利用增加幂次积分方法, 对高阶非线性系统的输出反馈, 跟踪等问题的研究均取得了很多有意义的成果[9–13].

值得指出的是, 目前已有的结果中, 均对系统非线性项$g_i(t,x,u,\theta)$有严格的限制条件, 即

$$|g_i(\cdot)| \leqslant \sum_{j=0}^{p_i-1} |x_{i+1}^j| \hat{g}_{i,j}(x_1,\cdots,x_i,\theta), \tag{2}$$

其中$\hat{g}_{i,j}(\cdot)$为非负的连续可微函数并且满足$\hat{g}_{i,j}(0,\cdots,0,\theta)=0$. 而当$\hat{g}_{i,j}(0,\cdots,0,\theta)\neq 0$时, 系统满足

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$u(0)=0$的连续状态反馈控制器则不一定存在, 如系统$\dot{x}=u^3+u^2+x$和$\dot{x}=u^3+u+x$. 第1个系统中$\hat{g}_{1,2}(\cdot)$恒为1, 第2个系统中$\hat{g}_{1,1}(\cdot)$恒为1, 显然是不满足上述条件的. 对第1个系统, 文献[3]证明了任意满足$u(0)=0$的连续控制器都不能使该系统稳定, 但第2个存在, 如取u满足$u^3+u=-2x$.

本文改进了增加幂次积分方法, 通过引入一列非齐次的辅助函数, 将传统的增加幂次积分方法改进为增加一个关于这种非齐次辅助函数的积分项的方法; 并利用该方法处理$\hat{g}_{i,j}(0,\cdots,0,\theta)\neq 0$的高阶非线性系统, 给出了这种系统满足$u(0)=0$的连续反馈自适应控制器存在的一个充分条件及其设计方法.

2　问题的描述与预备知识(Problem statement and preliminaries)

首先考虑仅在$j=q_i$为正奇数时, $\hat{g}_{i,j}(\cdot)$不满足前述条件的情况, 设$\hat{g}_{i,j}(0,\cdots,0,\theta)=c_i(\theta)\neq 0$, 则系统(1)可以改写为

$$\begin{cases}\dot{x}_i=d_i(\cdot)x_{i+1}^{p_i}+c_i(\cdot)x_{i+1}^{q_i}+f_i(t,x,u,\theta),\\ \dot{x}_n=d_n(\cdot)u^{p_n}+c_n(\cdot)u_n^{q_n}+f_n(t,x,u,\theta),\end{cases}\tag{3}$$

其中: $f_i(\cdot)=g_i(\cdot)-c_i(\cdot)x_{i+1}^{q_i}$; $d_i(\cdot),c_i(\cdot)$为连续函数, 它们可以是系统未知的时变参数, 为了讨论其自适应控制问题, 做如下假设:

假设 1　对$i=1,\cdots,n$, $1<q_i<p_i$均为奇正整数, 存在已知C^1函数$\hat{f}_{i,j}(x_1,\cdots,x_i,\theta)\geqslant 0$使得

$$|f_i(\cdot)|\leqslant\sum_{j=0}^{p_i-1}\hat{f}_{i,j}(x_1,\cdots,x_i,\theta)|x_{i+1}|^j,$$

其中$\hat{f}_{i,j}(0,\cdots,0,\theta)=0$.

假设 2　对$i=1,\cdots,n$, $c_i(\cdot),d_i(\cdot)$符号相同, 并且存在未知的正常数a_i,b_i和已知的光滑函数$\lambda_i(x_1,\cdots,x_i)>0$, $\mu_i(x_1,\cdots,x_i)>0$, 使得

$$0<a_i\lambda_i(\cdot)\leqslant|d_i(\cdot)|,|c_i(\cdot)|\leqslant b_i\mu_i(\cdot).$$

从假设2容易看到, $c_i(\cdot),d_i(\cdot)$是不变号的, 不失一般性, 本文后面的步骤中假设它们为正的.

注 1　与文献[1–4]等目前已有的结论相比, 系统(3)增加了一项$c_i(\cdot)x_{i+1}^{q_i}$, 方程右端的$d_i(\cdot)x_{i+1}^{p_i}+c_i(\cdot)x_{i+1}^{q_i}$体现出了明显的非齐次性, 所以本文把系统(3)称为非齐次高阶非线性系统. 由$c_i(\cdot)\neq 0$, 故该系统的非线性部分$c_i(\cdot)x_{i+1}^{q_i}+f_i(\cdot)$不满足文献[1–4]中增加幂次积分方法的基本假设(2), 因此, 该方法不能直接用于这种系统. 由前面的例子知, 当q_i为偶数时, 系统(3) 满足$u(0)=0$的连续反馈控制器可能不存在, 故假设1要求q_i为奇数.

下面给出本文用到的几个结论.

定理 1　对非自治系统

$$\dot{x}=f(t,x),\ t\in\mathbb{R},\ x\in\mathbb{R}^n,$$

其中$f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$是关于$(t,x)$的连续函数, 且$f(t,0)\equiv 0$. 若存在$C^1$函数$V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, 正定连续的径向无界函数$U_1:\mathbb{R}^n\to\mathbb{R}$, 和正定的连续函数$U_i:\mathbb{R}^n\to\mathbb{R}$, $i=2,3$, 使得

$$U_1(x)\leqslant V(t,x)\leqslant U_2(x),$$
$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\leqslant -U_3(x),$$

则该系统的平凡解$x=0$是全局强稳定的[14].

引理 1　对实值连续函数$f(x,y)$, 其中$x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$, 存在光滑函数$a(x)\geqslant 0$, $b(y)\geqslant 0$, $c(x)\geqslant 1$, $d(y)\geqslant 1$, 使得

$$|f(x,y)|\leqslant a(x)+b(y),|f(x,y)|\leqslant c(x)d(y).$$

引理 2　设c,d为实数, $\gamma(x,y)>0$为实值函数, 则

$$|x|^c|y|^d\leqslant\frac{c}{c+d}\gamma(x,y)|x|^{c+d}+\frac{d}{c+d}\gamma^{-\frac{c}{d}}(x,y)|y|^{c+d}.$$

引理 3　存在光滑函数$\bar{f}_i(x_1,\cdots,x_i)\geqslant 1$和依赖于$\theta$, a_i的未知常数$\varTheta_1\geqslant 1$, 使得

$$|f_i(\cdot)|\leqslant\frac{d_i(\cdot)}{2}|x_{i+1}^{p_i}|+\bar{f}_i(\cdot)\varTheta_1\sum_{l=1}^{i}|x_l|.$$

这3个引理及其证明可以在文献[4,15–16]中找到.

然后引入如下形式的辅助函数:

$$y_1(x)=x,\ y_i(x)=y_{i-1}(x^{p_{i-1}}+x^{q_{i-1}}).$$

利用二项式定理容易得到, 当$i=2,\cdots,n+1$时, $y_i(x)=x^{p_1\cdots p_{i-1}}+\cdots+x^{q_1\cdots q_{i-1}}$. 显然, 它们是严格单调递增的奇函数, 故其反函数$y_i^{-1}(x)$存在. 由$y_i(x)$的定义可以得到, 当$l\geqslant 1$时, $\lim\limits_{x\to 0}\dfrac{x^l}{y_k^{-1}(x)}=0$, 故在可去间断点$x=0$处, 定义$x^l/y_k^{-1}(x)=0$, 并且$x^l/y_k^{-1}(x)$在$[0,+\infty)$上是严格单调递增的. 对于这样定义的辅助函数, 有如下引理:

引理 4　对任意实数x_1,x_2, 有

$$y_i^{-1}(|x_1|+|x_2|)\leqslant y_i^{-1}(|x_1|)+y_i^{-1}(|x_2|),$$
$$2y_i^{-1}(|x_1+x_2|)\geqslant|y_i^{-1}(x_1)+y_i^{-1}(x_2)|.$$

引理 5　若$|x_1|\geqslant 1$, 有

$$y_i(|x_1x_2|)\geqslant|x_1|y_i(|x_2|),$$
$$y_i^{-1}(|x_1x_2|)\leqslant|x_1|y_i^{-1}(|x_2|).$$

引理 6　对任意实数x_1,x_2, 有

$$y_i(|x_1x_2|)\leqslant|y_i(x_1)y_i(x_2)|,$$
$$y_i^{-1}(|x_1x_2|)\geqslant|y_i^{-1}(x_1)y_i^{-1}(x_2)|.$$

利用$y_i(\cdot)$和$y_i^{-1}(\cdot)$的定义, 这3个引理的证明非常简单, 故本文省略了这些证明.

引理 7 如下不等式成立:

$$\frac{1}{p_1\cdots p_{i-1}}(y_i(x))' \leqslant \frac{y_i(x)}{x}, \tag{4}$$

$$[\frac{x^2}{y_i^{-1}(x)}]' \leqslant (2-\frac{1}{p_1\cdots p_{i-1}})\frac{x}{y_i^{-1}(x)}, \tag{5}$$

$$\frac{x_1+x_2}{y_i^{-1}(x_1+x_2)} \leqslant \frac{x_1}{y_i^{-1}(x_1)}+\frac{x_2}{y_i^{-1}(x_2)}. \tag{6}$$

证 由$y_i(\cdot)$的定义和二项式定理,

$$y_i(x)=x^{p_1\cdots p_{i-1}}+\sum_k m_k x^{n_k}+x^{q_1\cdots q_{i-1}},$$

其中m_k, n_k由二项式定理确定, 且$p_1\cdots p_{i-1}>n_k>q_1\cdots q_{i-1}$均为奇数, 求导即可得到式(4)成立.

由反函数求导公式和式(4)有

$$(y_i^{-1}(x))' \geqslant \frac{1}{p_1\cdots p_{i-1}}\frac{y_i^{-1}(x)}{x},$$

从而

$$[\frac{x^2}{y_i^{-1}(x)}]'=\frac{2x}{y_i^{-1}(x)}-\frac{x^2(y_i^{-1}(x))'}{(y_i^{-1}(x))^2}\leqslant (2-\frac{1}{p_1\cdots p_{i-1}})\frac{x}{y_i^{-1}(x)},$$

即式(5)成立.

对不等式(6), 若x_1, x_2有一个为零时显然成立; 若x_1, x_2均不为零, 由$x/y_i^{-1}(x)>0$及$y_i^{-1}(x)$严格单调递增, 故x_1, x_2同号时,

$$|y_i^{-1}(x_1+x_2)|>|y_i^{-1}(x_1)|,$$

且$|y_i^{-1}(x_1+x_2)|>|y_i^{-1}(x_2)|$, 此时显然有式(6)成立, 当$x_1, x_2$异号时, 不妨设$x_1\geqslant |x_2|=-x_2$, 则$x_1>x_1+x_2\geqslant 0$, 由于$x/y_i^{-1}(x)$在$[0,+\infty)$上严格单调递增, 有

$$\frac{x_1}{y_i^{-1}(x_1)}>\frac{x_1+x_2}{y_i^{-1}(x_1+x_2)},$$

从而式(6)成立.

引理 8 对任意$l\leqslant k$, 存在光滑函数$\alpha_{k,l}(x)>0$, 使得$y_l^{-1}(|x|)\leqslant \alpha_{k,l}(x)y_k^{-1}(|x|)$.

证 显然, 当$k=1$时, 上式成立. 设$k-1$时成立, 即存在光滑函数$\alpha_{k-1,l}(x)>0$, 使得

$$y_l^{-1}(|x|)\leqslant \alpha_{k-1,l}(x)y_{k-1}^{-1}(|x|),\ l\leqslant k-1.$$

则在k时, 由$y_k(x)=y_{k-1}(x^{p_{k-1}}+x^{q_{k-1}})$, 故

$$|x|=y_{k-1}((y_k^{-1}(|x|))^{p_{k-1}}+(y_k^{-1}(|x|))^{q_{k-1}}),$$

用y_{k-1}^{-1}作用有

$$y_{k-1}^{-1}(|x|)=(y_k^{-1}(|x|))^{p_{k-1}}+(y_k^{-1}(|x|))^{q_{k-1}},$$

由$y_k(\cdot)$的定义容易得到$y_k^{-1}(|x|)\leqslant |x|^{\frac{1}{p_1\cdots p_{k-1}}}$, 故存在光滑函数$\psi_1(x)>0$(如取$\psi_1(x)=1+x^2$), 使得

$$(y_k^{-1}(|x|))^{p_{k-1}-1}\leqslant |x|^{\frac{1}{p_1\cdots p_{k-2}}-\frac{1}{p_1\cdots p_{k-1}}}<\psi_1(x).$$

同样的, 由$y_k^{-1}(|x|)\leqslant |x|^{\frac{1}{q_1\cdots q_{k-1}}}$, 因此存在光滑函数$\psi_2(x)>0$, 使得$(y_k^{-1}(|x|))^{q_{k-1}-1}<\psi_2(x)$, 从而有

$$y_{k-1}^{-1}(|x|)<[\psi_1(x)+\psi_2(x)]y_k^{-1}(|x|).$$

取$\alpha_{k,l}(x)=\alpha_{k-1,l}(x)[\psi_1(x)+\psi_2(x)]>0$, 其中$l\leqslant k-1$和$\alpha_{k,k}(x)=1$, 有引理8成立.

3 自适应控制器的设计(Design of adaptive controllers)

取

$$\Theta=\max_{3\leqslant k\leqslant n}\{\Theta_2^2,\frac{1}{a^2},\frac{\Theta_2}{a},\frac{1}{a^2}\times(ab_{k-1})^{2p_1\cdots p_{k-2}},\frac{1}{a^2}\times(ab_{k-1})^{2q_1\cdots q_{k-2}},\Theta_2^{\frac{2p_1\cdots p_{k-1}}{2p_1\cdots p_{k-1}-1}}\times a^{\frac{2-2p_1\cdots p_{k-1}}{2p_1\cdots p_{k-1}-1}},\Theta_2^{\frac{2q_1\cdots q_{k-1}}{2q_1\cdots q_{k-1}-1}}\times a^{\frac{2-2q_1\cdots q_{k-1}}{2q_1\cdots q_{k-1}-1}}\}$$

为一个未知的参数, 其中:

$$a=\min\{a_1,\cdots,a_n\},\ \Theta_2=\max\{b_1,\cdots,b_{n-1},\Theta_1\}.$$

记$\hat{\Theta}(t)$为Θ的估计, $\tilde{\Theta}(t)=\Theta-\hat{\Theta}(t)$为误差.

下面给出系统(3)的连续自适应控制器的递推设计方法.

步骤 1 对正定、连续可微且径向无界的Lyapunov函数$V_1(x_1,\tilde{\Theta})=\frac{1}{2}x_1^2+\frac{a}{2}\tilde{\Theta}^2$有

$$\dot{V}_1=x_1[d_1(\cdot)x_2^{p_1}+c_1(\cdot)x_2^{q_1}+f_1(\cdot)]-a\tilde{\Theta}\dot{\hat{\Theta}}.$$

由引理3,

$$\dot{V}_1\leqslant x_1[d_1(\cdot)x_2^{p_1}+c_1(\cdot)x_2^{q_1}]+\frac{d_1(\cdot)}{2}|x_1x_2^{p_1}|+x_1^2\bar{f}_1(x_1)\Theta_1-a\tilde{\Theta}\dot{\hat{\Theta}}.$$

取虚拟控制量x_2^*满足

$$y_2(x_2^*)=-x_1\frac{2(\bar{f}_1(x_1)+n+\frac{1}{4})\sqrt{1+\hat{\Theta}^2}}{\lambda_1(x_1)}:=-x_1\beta_1(x_1,\hat{\Theta}),$$

其中$\beta_1(\cdot)>0$为光滑函数, 则有

$$\begin{aligned}\dot{V}_1\leqslant{}& x_1\left[d_1(\cdot)\left(x_2^{p_1}-x_2^{*p_1}\right)+c_1(\cdot)\left(x_2^{q_1}-x_2^{*q_1}\right)\right]+\\&d_1(\cdot)x_1x_2^{*p_1}+c_1(\cdot)x_1x_2^{*q_1}+\\&\frac{d_1(\cdot)}{2}|x_1x_2^{p_1}|+x_1^2\bar{f}_1(\cdot)\Theta_1-a\tilde{\Theta}\dot{\hat{\Theta}}\leqslant\\&\frac{3b_1\mu_1(\cdot)}{2}|x_1(x_2^{p_1}+x_2^{q_1}-x_2^{*p_1}-x_2^{*q_1})|+\\&\frac{d_1(\cdot)x_1x_2^{*p_1}}{2}+\frac{c_1(\cdot)x_1x_2^{*q_1}}{2}+\\&x_1^2\bar{f}_1(\cdot)\Theta_1-a\tilde{\Theta}\dot{\hat{\Theta}}.\end{aligned}$$

设$\tau_1(x_1)=\bar{f}_1(x_1)+n+\frac{1}{4}$, $\rho_1(x_1)=x_1^2\tau_1(x_1)$. 由$x_1y_2(x_2^*)\leqslant 0$, 容易有$x_1x_2^{*p_1}\leqslant 0$和$x_1x_2^{*q_1}\leqslant 0$,

故由Θ的定义可以得到

$$\frac{d_1(\cdot)x_1x_2^{*p_1}}{2}+\frac{c_1(\cdot)x_1x_2^{*q_1}}{2}+x_1^2\bar{f}_1(\cdot)\Theta_1-a\tilde{\Theta}\dot{\hat{\Theta}}\leqslant$$
$$ax_1\frac{\lambda_1(\cdot)(x_2^{*p_1}+x_2^{*q_1})}{2}+ax_1^2\bar{f}_1(\cdot)\Theta-a\tilde{\Theta}\dot{\hat{\Theta}}\leqslant$$
$$ax_1(\frac{\lambda_1(\cdot)y_2(x_2^*)}{2}+x_1\tau_1(\cdot)\hat{\Theta})-\frac{n}{a}x_1^2+$$
$$a\tilde{\Theta}[\rho_1(\cdot)-\dot{\hat{\Theta}}]\leqslant$$
$$-\frac{n}{a}x_1^2+a\tilde{\Theta}[\rho_1(\cdot)-\dot{\hat{\Theta}}],$$

从而有

$$\dot{V}_1\leqslant\frac{3b_1\mu_1(\cdot)}{2}|x_1(x_2^{p_1}+x_2^{q_1}-x_2^{*p_1}-x_2^{*q_1})|-\frac{n}{a}x_1^2+a\tilde{\Theta}[\rho_1(\cdot)-\dot{\hat{\Theta}}].$$

步骤 2 设第$k-1$步存在正定, 连续可微且径向无界的Lyapunov函数$V_{k-1}(x_1,\cdots,x_{k-1},\tilde{\Theta})$, 光滑函数$\tau_1(x_1),\cdots,\tau_{k-1}(x_1,\cdots,x_{k-1})$; $\rho_1(x_1),\cdots,\rho_{k-1}(x_1,\cdots,x_{k-1})$, 和连续虚拟控制器$x_1^*,\cdots,x_k^*$满足

$$\begin{cases}y_1(x_1^*)=0,\\ y_2(x_2^*)=-\xi_1\beta_1(\cdot),\\ \quad\vdots\\ y_k(x_k^*)=-\xi_{k-1}\beta_{k-1}(\cdot),\end{cases}$$
$$\begin{cases}\xi_1=x_1-x_1^*,\\ \xi_2=y_2(x_2)-y_2(x_2^*),\\ \quad\vdots\\ \xi_k=y_k(x_k)-y_k(x_k^*),\end{cases}$$

其中$\beta_i(x_1,\cdots,x_i,\hat{\Theta})>0$为光滑函数, 使得

$$\dot{V}_{k-1}\leqslant-\frac{n-k+2}{a}\sum_{l=1}^{k-1}\xi_l^2+a\tilde{\Theta}[\rho_{k-1}(\cdot)-\dot{\hat{\Theta}}]-\sum_{i=1}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}(\rho_{k-1}(\cdot)-\dot{\hat{\Theta}})+\frac{3}{2}b_{k-1}\mu_{k-1}(\cdot)|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}|\times|(x_k^{p_{k-1}}+x_k^{q_{k-1}}-x_k^{*p_{k-1}}-x_k^{*q_{k-1}})|.\tag{7}$$

则在第k步时, 对于Lyapunov函数

$$V_k(x_1,\cdots,x_k,\hat{\Theta})=V_{k-1}(\cdot)+W_k(x_1,\cdots,x_k,\hat{\Theta}),$$
$$W_k(x_1,\cdots,x_k,\hat{\Theta})=\int_{x_k^*}^{x_k}\frac{(y_k(s)-y_k(x_k^*))^2}{y_k^{-1}(y_k(s)-y_k(x_k^*))}\mathrm{d}s.$$

显然, $W_k(\cdot)\geqslant0$为连续可微函数且

$$\frac{\partial W_k}{\partial x_k}=\frac{\xi_k^2}{y_k^{-1}(\xi_k)},$$
$$\frac{\partial W_k}{\partial x_l}=-\frac{\partial y_k(x_k^*)}{\partial x_l}\int_{x_k^*}^{x_k}\frac{\partial[\frac{(y_k(s)-y_k(x_k^*))^2}{y_k^{-1}(y_k(s)-y_k(x_k^*))}]}{\partial(y_k(s)-y_k(x_k^*))}\mathrm{d}s,$$
$$\frac{\partial W_k}{\partial\hat{\Theta}}=-\frac{\partial y_k(x_k^*)}{\partial\hat{\Theta}}\int_{x_k^*}^{x_k}\frac{\partial[\frac{(y_k(s)-y_k(x_k^*))^2}{y_k^{-1}(y_k(s)-y_k(x_k^*))}]}{\partial(y_k(s)-y_k(x_k^*))}\mathrm{d}s,$$

其中$1\leqslant l\leqslant k-1$.

由于$y_k^{-1}(|x|)\leqslant|x|^{\frac{1}{p_1\cdots p_{k-1}}}$, 故当$x_k\geqslant x_k^*$时, 有

$$W_k(\cdot)\geqslant\int_{x_k^*}^{x_k}(y_k(s)-y_k(x_k^*))^{2-\frac{1}{p_1\cdots p_{k-1}}}\mathrm{d}s\geqslant$$
$$\int_{x_k^*}^{x_k}(s^{p_1\cdots p_{k-1}}-x_k^{*p_1\cdots p_{k-1}})^{2-\frac{1}{p_1\cdots p_{k-1}}}\mathrm{d}s\geqslant$$
$$\int_{x_k^*}^{x_k}2^{-p_1\cdots p_{k-1}+1}(s-x_k^*)^{2p_1\cdots p_{k-1}-1}\mathrm{d}s=$$
$$h_k(x_k-x_k^*)^{2p_1\cdots p_{k-1}},$$

其中$h_k>0$为常数. 显然, $x_k\leqslant x_k^*$时上式也成立, 即

$$V_k\geqslant V_{k-1}+c_k(x_k-x_k^*)^{2p_1\cdots p_{k-1}},$$

故$V_k(\cdot)$是正定且径向无界的.

显然

$$\dot{V}_k(\cdot)=\dot{V}_{k-1}+\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l+\frac{\partial W_k}{\partial\hat{\Theta}}\dot{\hat{\Theta}}+\frac{\xi_k^2}{y_k^{-1}(\xi_k)}[d_k(\cdot)x_{k+1}^{p_k}+c_k(\cdot)x_{k+1}^{q_k}+f_k(\cdot)].\tag{8}$$

并且根据附录, 存在光滑函数$\gamma_{k,i}(x_1,\cdots,x_k,\hat{\Theta})\geqslant0(i=1,2,3)$使得如下不等式成立:

$$\frac{3}{2}b_{k-1}\mu_{k-1}(\cdot)|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}|\times|(x_k^{p_{k-1}}+x_k^{q_{k-1}}-x_k^{*p_{k-1}}-x_k^{*q_{k-1}})|\leqslant\frac{\xi_{k-1}^2}{4a}+a\Theta\gamma_{k,1}(\cdot)\xi_k^2,\tag{9}$$

$$\frac{\xi_k^2}{y_k^{-1}(\xi_k)}f_k(\cdot)\leqslant\frac{d_k(\cdot)}{2}|\frac{\xi_k^2}{y_k^{-1}(\xi_)}x_{k+1}^{p_k}|+\frac{1}{4a}\sum_{l=1}^{k-1}\xi_l^2+a\Theta\gamma_{k,2}(\cdot)\xi_k^2,\tag{10}$$

$$\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l\leqslant a\Theta\gamma_{k,3}(\cdot)\xi_k^2+\frac{1}{4a}\sum_{l=1}^{k-1}\xi_l^2.\tag{11}$$

将式(7)(9)–(11)代入式(8)得

$$\dot{V}_k\leqslant-\frac{n-k+\frac{5}{4}}{a}\sum_{l=1}^{k}\xi_l^2+\frac{n-k+\frac{5}{4}}{a}\xi_k^2+a\Theta[\gamma_{k,1}(\cdot)+\gamma_{k,2}(\cdot)+\gamma_{k,3}(\cdot)]\xi_k^2+\frac{d_k(\cdot)}{2}|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}x_{k+1}^{p_k}|+a\tilde{\Theta}[\rho_{k-1}(\cdot)-\dot{\hat{\Theta}}]-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}(\rho_{k-1}(\cdot)-\dot{\hat{\Theta}})+\frac{\partial W_k}{\partial\hat{\Theta}}\dot{\hat{\Theta}}+\frac{\xi_k^2}{y_k^{-1}(\xi_k)}[d_k(\cdot)x_{k+1}^{p_k}+c_k(\cdot)x_{k+1}^{q_k}].\tag{12}$$

注意到$a\Theta \geqslant \frac{1}{a}$, 并取

$$\tau_k(x_1,\cdots,x_k,\hat\Theta)=\gamma_{k,1}(\cdot)+\gamma_{k,2}(\cdot)+\gamma_{k,3}(\cdot)+n-k+\frac{5}{4},$$

$$\rho_k(x_1,\cdots,x_k,\hat\Theta)=\rho_{k-1}(\cdot)+\tau_k(\cdot)\xi_k^2=\sum_{i=1}^{k}\tau_i(\cdot)\xi_i^2,$$

代入式(12)有

$$\begin{aligned}\dot V_k\leqslant&-\frac{n-k+\frac{5}{4}}{a}\sum_{l=1}^{k}\xi_l^2+a\hat\Theta\tau_k(\cdot)\xi_k^2+\\&a\tilde\Theta[\rho_k(\cdot)-\dot{\hat\Theta}]-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat\Theta}(\rho_{k-1}(\cdot)-\dot{\hat\Theta})+\\&\frac{\partial W_k}{\partial\hat\Theta}\dot{\hat\Theta}+\frac{\xi_k^2}{y_k^{-1}(\xi_k)}[d_k(\cdot)x_{k+1}^{p_k}+c_k(\cdot)x_{k+1}^{q_k}]+\\&\frac{d_k(\cdot)}{2}\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}x_{k+1}^{p_k}\Big|,\end{aligned}\tag{13}$$

其中

$$\begin{aligned}&-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat\Theta}(\rho_{k-1}(\cdot)-\dot{\hat\Theta})+\frac{\partial W_k}{\partial\hat\Theta}\dot{\hat\Theta}=\\&-\sum_{i=2}^{k}\frac{\partial W_i}{\partial\hat\Theta}(\rho_k(\cdot)-\dot{\hat\Theta})+\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat\Theta}\tau_k(\cdot)\xi_k^2+\\&\frac{\partial W_k}{\partial\hat\Theta}\rho_k(\cdot).\end{aligned}\tag{14}$$

由附录, 存在光滑函数$\gamma_{k,4}(x_1,\cdots,x_k,\hat\Theta)\geqslant 0$, 使得

$$\begin{aligned}&\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat\Theta}\tau_k(\cdot)\xi_k^2+\frac{\partial W_k}{\partial\hat\Theta}\rho_k(\cdot)\leqslant\\&a\gamma_{k,4}(\cdot)\xi_k^2+\frac{1}{4a}\sum_{l=1}^{k}\xi_l^2.\end{aligned}\tag{15}$$

将式(14)–(15)代入式(13) 得

$$\begin{aligned}\dot V_k\leqslant&-\frac{n-k+1}{a}\sum_{l=1}^{k}\xi_l^2+a\hat\Theta\tau_k(\cdot)\xi_k^2+\\&a\tilde\Theta[\rho_k(\cdot)-\dot{\hat\Theta}]-\sum_{i=2}^{k}\frac{\partial W_i}{\partial\hat\Theta}(\rho_k(\cdot)-\dot{\hat\Theta})+\\&a\gamma_{k,4}(\cdot)\xi_k^2+\frac{\xi_k^2}{y_k^{-1}(\xi_k)}[d_k(\cdot)x_{k+1}^{p_k}+c_k(\cdot)x_{k+1}^{q_k}]+\\&\frac{d_k(\cdot)}{2}\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}x_{k+1}^{p_k}\Big|.\end{aligned}\tag{16}$$

取连续虚拟控制器x_{k+1}^*满足

$$y_{k+1}(x_{k+1}^*)=-\xi_k\beta_k(x_1,\cdots,x_k,\hat\Theta),$$

其中

$$\beta_k(\cdot)=y_k\Big[\frac{2(\tau_k(\cdot)\sqrt{1+\hat\Theta^2}+\gamma_{k,4}(\cdot))}{\lambda_k(\cdot)}\Big]>0$$

为光滑函数. 由于

$$\begin{aligned}&\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{*p_k}+x_{k+1}^{*q_k})=-\frac{\xi_k^2}{y_k^{-1}(\xi_k)}y_k^{-1}(\xi_k\beta_k(\cdot))\leqslant\\&-\frac{\xi_k^2}{y_k^{-1}(\xi_k)}y_k^{-1}(\xi_k)y_k^{-1}(\beta_k(\cdot))=-\xi_k^2y_k^{-1}(\beta_k(\cdot))\leqslant 0,\end{aligned}$$

故由$\beta_k(\cdot)$的定义有

$$\begin{aligned}&a\hat\Theta\tau_k(\cdot)\xi_k^2+a\gamma_{k,4}(\cdot)\xi_k^2<a\frac{\lambda_k(\cdot)}{2}y_k^{-1}(\beta_k(\cdot))\xi_k^2\leqslant\\&-a\frac{\lambda_k(\cdot)}{2}\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{*p_k}+x_{k+1}^{*q_k})\end{aligned}\tag{17}$$

和

$$\begin{aligned}&\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(d_k(\cdot)x_{k+1}^{p_k}+c_k(\cdot)x_{k+1}^{q_k})+\\&\frac{d_k(\cdot)}{2}\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}x_{k+1}^{p_k}\Big|\leqslant\\&b_k\mu_k(\cdot)\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{p_k}+x_{k+1}^{q_k}-x_{k+1}^{*p_k}-x_{k+1}^{*q_k})\Big|+\\&\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(d_k(\cdot)x_{k+1}^{*p_k}+c_k(\cdot)x_{k+1}^{*q_k})+\\&\frac{d_k(\cdot)}{2}\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}x_{k+1}^{p_k}\Big|\leqslant\\&\frac{3b_k\mu_k(\cdot)}{2}\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{p_k}+x_{k+1}^{q_k}-x_{k+1}^{*p_k}-x_{k+1}^{*q_k})\Big|+\\&\frac{\xi_k^2}{y_k^{-1}(\xi_k)}\Big(\frac{d_k(\cdot)}{2}x_{k+1}^{*p_k}+\frac{c_k(\cdot)}{2}x_{k+1}^{*q_k}\Big)\leqslant\\&\frac{3b_k\mu_k(\cdot)}{2}\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{p_k}+x_{k+1}^{q_k}-x_{k+1}^{*p_k}-x_{k+1}^{*q_k})\Big|+\\&\frac{a\lambda_k(\cdot)}{2}\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{*p_k}+x_{k+1}^{*q_k}).\end{aligned}\tag{18}$$

将式(17)–(18)代入式(16)得

$$\begin{aligned}\dot V_k(\cdot)\leqslant&-\frac{n-k+1}{a}\sum_{l=1}^{k}\xi_l^2+a\tilde\Theta[\rho_k(\cdot)-\dot{\hat\Theta}]-\\&\sum_{i=2}^{k}\frac{\partial W_i}{\partial\hat\Theta}(\rho_k(\cdot)-\dot{\hat\Theta})+\frac{3b_k\mu_k(\cdot)}{2}\cdot\\&\Big|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}(x_{k+1}^{p_k}+x_{k+1}^{q_k}-x_{k+1}^{*p_k}-x_{k+1}^{*q_k})\Big|,\end{aligned}$$

即式(7)在k时也成立.

这样, 按照递推的方法, 在第n步取$u=x_{n+1}^*$, $\dot{\hat\Theta}=\rho_n(\cdot)$, 就有

$$\dot V_n(\cdot)\leqslant-\frac{1}{a}\sum_{l=1}^{n}\xi_l^2,$$

其中$V_n(\cdot)$为正定的连续可微且径向无界的函数, 从而, 由定理1容易得到:

定理 2 对含有未知参数的高阶非线性系统(3), 如果它满足假设1–2, 则存在连续控制器$u=u(x)$满足$u(0)=0$, 使得式(3)的平凡解$x=0$是全局强稳定的.

该定理的结论很容易推广到更一般的情况, 即有多个$\hat g_{i,j}(0,\cdots,0,\theta)=c_{i,j}(\cdot)\neq 0$和$p_i,q_i$非整数的情

况, 即系统

$$\begin{cases}\dot{x}_i = d_i(\cdot)x_{i+1}^{p_i} + \sum\limits_{j=1}^{l_i} c_{i,j}(\cdot)x_{i+1}^{q_{i,j}} + f_i(\cdot),\\ \dot{x}_n = d_n(\cdot)u^{p_n} + \sum\limits_{j=1}^{l_n} c_{n,j}(\cdot)u^{q_{n,j}} + f_n(\cdot),\end{cases} \tag{19}$$

其中: l_i为有限正实数, $p_i, q_i \in \{p/q|p, q$为正的奇整数$\}$, 且$1 < q_{i,j} < p_i$. $d_i(\cdot), c_{i,j}(\cdot), f_i(\cdot)$为连续函数且满足:

假设 3 对$i = 1, \cdots, n$, 存在已知连续可微函数$\hat{f}_{i,j}(x_1, \cdots, x_i, \theta) \geqslant 0$使得

$$|f_i(\cdot)| \leqslant \sum_{j=0}^{l_i} \hat{f}_{i,j}(x_1, \cdots, x_i, \theta)|x_{i+1}|^{q_{i,j}},$$

其中$\hat{f}_{i,j}(0, \cdots, 0, \theta) = 0, q_{i,0} = 0, x_{n+1} = u$.

假设 4 对$i = 1, \cdots, n, j = 1, \cdots, l_i$; $c_{i,j}(\cdot)$恒为零或者与$d_i(\cdot)$同号, 并且存在已知的光滑函数$\lambda_i(x_1, \cdots, x_i) > 0, \mu_i(x_1, \cdots, x_i) > 0$和未知正常数$a_i, b_i$, 使得

$$0 < a_i\lambda_i(\cdot) \leqslant |d_i(\cdot)|, |c_{i,j}(\cdot)| \leqslant b_i\mu_i(\cdot).$$

对系统(19), 只要取辅助函数

$$y_1(x) = x, y_{i+1}(x) = y_i(x^{p_i} + \sum_{j=1}^{l_i} \varepsilon_{i,j}x^{q_{i,j}}),$$

其中: 当$c_{i,j}(\cdot)$不恒等于零时, $\varepsilon_{i,j} = 1$, 而$c_{i,j}(\cdot)$恒等于零时, $\varepsilon_{i,j} = 0$. 注意到前面给出的引理对这样定义的辅助函数仍然成立, 因此, 利用定理2的方法, 容易证明:

定理 3 对含未知参数的高阶非线性系统(19), 如果它满足假设3–4, 则存在连续控制器$u = u(x)$满足$u(0) = 0$, 使得式(19)的平凡解$x = 0$是全局强稳定的.

注 2 注意到若式(19)中所有的$c_{ij}(\cdot)$均为零, 则系统(19)退化为文献[1–4]研究的满足条件(2)的系统, 此时, 本文中给出的辅助函数变为$y_i(x) = x^{p_1 \cdots p_{i-1}}$, 因此, 文献[1–4]中给出的增加幂次积分方法及其结果均可归结为本文的一个特例.

4 仿真例子(Simulation example)

考虑如下系统:

$$\begin{cases}\dot{x}_1 = \theta_{11}(5+\sin t)x_2^{\frac{7}{3}} + 4\theta_{12}x_2^{\frac{5}{3}} + \theta_{13}x_1x_2 + \theta_{14}x_1,\\ \dot{x}_2 = 4\theta_{21}u^5 + 5\theta_{22}u^{\frac{5}{3}} + \theta_{23}x_2,\end{cases} \tag{20}$$

其中: $\theta_{i,j}$为未知常数, θ_{11}和θ_{12}同号, θ_{21}和θ_{22}同号. 取辅助函数

$$y_2(x) = x^{\frac{7}{3}} + x^{\frac{5}{3}}, y_3(x) = y_2(x^5 + x^{\frac{5}{3}}),$$

取

$$\tau_1(\cdot) = 3.25 + \sqrt{1 + x_1^2},\ \beta_1 = 0.1\tau_1(\cdot)\sqrt{1 + \hat{\Theta}^2}$$

和虚拟控制器x_2^*满足$y_2(x_2^*) = -x_1\beta_1(x_1, \hat{\Theta})$, 根据前面的方法, 取

$$\gamma_{21}(\cdot) = 2.25, \gamma_{22}(\cdot) = 2.54(1 + \beta_1)^{10/7} + 1,$$

$$\gamma_{23}(\cdot) = 5.4(\beta_1(\cdot) + |0.1x_1\sqrt{1 + \hat{\Theta}^2}|) + [3.6(\beta_1(\cdot) + |0.1x_1\sqrt{1 + \hat{\Theta}^2}|) \times (1.5\beta_1 + 1 + \sqrt{1 + x_1^2})]^2,$$

$$\tau_2(\cdot) = \gamma_{21}(\cdot) + \gamma_{22}(\cdot) + \gamma_{23}(\cdot) + 1.25,$$

$$\xi_2 = y_2(x_2) - y_2(x_2^*),$$

$$\gamma_{24} = 0.13(\tau_1x_1)^2[(\tau_1x_1)^2 + (\tau_2\xi_2)^2],$$

$$\beta_2(\cdot) = y_2[0.1(\tau_2\sqrt{1 + \hat{\Theta}^2} + \gamma_{24})].$$

这样, 取$u = -y_3^{-1}(\xi_2\beta_2(\cdot))$, 则对Lyapunov函数

$$V(\cdot) = \frac{1}{2}x_1^2 + \frac{a}{2}\tilde{\Theta}^2 + \int_{x_2^*}^{x_2} \frac{(y_2(s) - y_2(x_2^*))^2}{y_2^{-1}(y_2(s) - y_2(x_2^*))}\mathrm{d}s,$$

有$\dot{V}(\cdot) \leqslant -\dfrac{1}{a}(x_1^2 + \xi_2^2)$, 其中$a > 0$. 图1–2给出了在$\theta_{i,j} = 1$, 初始值为$x_1(0) = 0.5, x_2(0) = -0.5, \hat{\Theta}(0) = 1$时, $x_1, x_2, \hat{\Theta}$的轨迹.

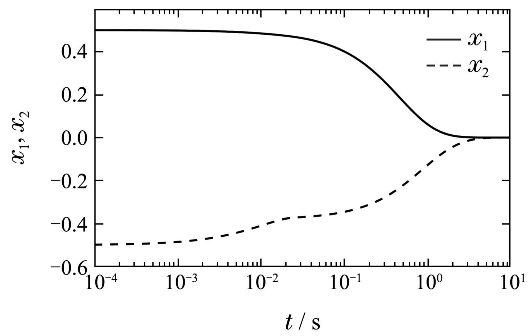

图 1 x_1, x_2的轨迹

Fig. 1 The trajectories of x_1, x_2

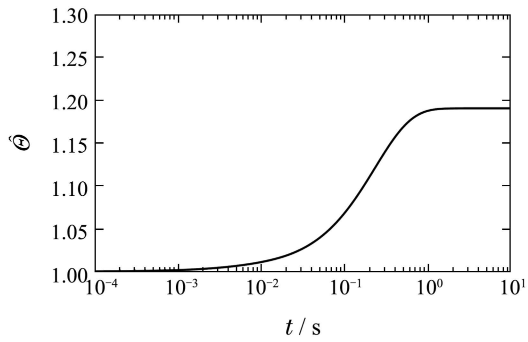

图 2 $\hat{\Theta}$的轨迹

Fig. 2 The trajectory of $\hat{\Theta}$

5 结论(Conclusions)

对于高阶非线性系统(1), 本文通过引入一列辅助函数, 将传统的增加幂次积分方法修改为增加关于辅助函数的积分项的方法, 放宽了目前已有结果[1–4]的限制条件, 给出了具有非齐次特性的高阶非线性系

统(3)(19)的满足$u(0)=0$的连续反馈控制器存在的一个充分条件及其设计方法.

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附录 不等式的证明 (Appendix Proof of inequalities)

A.1. 存在光滑函数$\gamma_{k,1}(\cdot)\geqslant 0$, 使得式(9)成立.

证 由引理4,

$$|x_k^{p_{k-1}}+x_k^{q_{k-1}}-x_k^{*p_{k-1}}-x_k^{*q_{k-1}}|=$$
$$|y_{k-1}^{-1}(y_k(x_k))-y_{k-1}^{-1}(y_k(x_k^*))|\leqslant$$
$$2y_{k-1}^{-1}(|y_k(x_k)-y_k(x_k^*)|)=2y_{k-1}^{-1}(\xi_k),$$

故

$$\frac{3}{2}b_{k-1}\mu_{k-1}(\cdot)|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}|\times$$
$$|(x_k^{p_{k-1}}+x_k^{q_{k-1}}-x_k^{*p_{k-1}}-x_k^{*q_{k-1}})|\leqslant$$
$$3b_{k-1}\mu_{k-1}|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}y_{k-1}^{-1}(\xi_k)|.$$

由$y_i(x)$的定义, 有$|y_{k-1}(x)|\geqslant|x|^{p_1\cdots p_{k-2}}$, 且存在常数$M_{k-1}$, 使得当$|x|\geqslant 1$时, $|y_{k-1}(x)|\leqslant M_{k-1}|x|^{p_1\cdots p_{k-2}}$. 故

$$|y_{k-1}^{-1}(x)|\leqslant|x|^{\frac{1}{p_1\cdots p_{k-2}}},\ |y_{k-1}^{-1}(x)|\geqslant\frac{1}{M_{k-1}}|x|^{\frac{1}{p_1\cdots p_{k-2}}}.$$

从而, 当$|\xi_{k-1}|\geqslant 1$时, 由引理2, 存在C^0函数$\phi_{k,1}(\cdot)$, 使得

$$3b_{k-1}\mu_{k-1}|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}y_{k-1}^{-1}(\xi_k)|\leqslant$$
$$3b_{k-1}\mu_{k-1}M_{k-1}|\xi_{k-1}^{2-\frac{1}{p_1\cdots p_{k-2}}}||\xi_k^{\frac{1}{p_1\cdots p_{k-2}}}|\leqslant$$
$$\frac{1}{4a}\xi_{k-1}^2+\phi_{k,1}(\cdot)\xi_k^2,$$

其中

$$\phi_{k,1}(\cdot)=\frac{1}{2p_1\cdots p_{k-2}}(3b_{k-1}\mu_{k-1}M_{k-1})^{2p_1\cdots p_{k-2}}\times$$
$$[\frac{1}{4a}\frac{2p_1\cdots p_{k-2}}{2p_1\cdots p_{k-2}-1}]^{-(2p_1\cdots p_{k-2}-1)}.$$

由$|y_{k-1}(x)|\geqslant|x|^{q_1\cdots q_{k-2}}$, 和$|x|\leqslant 1$时$|y_{k-1}(x)|\leqslant M_{k-1}|x|^{q_1\cdots q_{k-2}}$, 同理, 当$|\xi_{k-1}|\leqslant 1$时, 存在$C^0$函数$\phi_{k,2}(\cdot)$, 使得

$$\frac{3}{2}b_{k-1}\mu_{k-1}(\cdot)|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}|\times$$
$$|(x_k^{p_{k-1}}+x_k^{q_{k-1}}-x_k^{*p_{k-1}}-x_k^{*q_{k-1}})|\leqslant$$
$$\frac{1}{4a}\xi_{k-1}^2+\phi_{k,2}(\cdot)\xi_k^2,$$

其中

$$\phi_{k,2}(\cdot)=\frac{1}{2q_1\cdots q_{k-2}}(3b_{k-1}\mu_{k-1}M_{k-1})^{2q_1\cdots q_{k-2}}\times$$
$$[\frac{1}{4a}\frac{2q_1\cdots q_{k-2}}{2q_1\cdots q_{k-2}-1}]^{-(2q_1\cdots q_{k-2}-1)},$$

从而有

$$3b_{k-1}\mu_{k-1}|\frac{\xi_{k-1}^2}{y_{k-1}^{-1}(\xi_{k-1})}y_{k-1}^{-1}(\xi_k)|\leqslant$$
$$\frac{1}{4a}\xi_{k-1}^2+(\phi_{k,1}(\cdot)+\phi_{k,2}(\cdot))\xi_k^2.$$

由$\Theta\geqslant a^{2p_1\cdots p_{k-2}-2}b_{k-1}^{2p_1\cdots p_{k-2}}, a^{2q_1\cdots q_{k-2}-2}b_{k-1}^{2q_1\cdots q_{k-2}}$, 故

$$\phi_{k,1}(\cdot)\leqslant a\Theta\frac{1}{2p_1\cdots p_{k-2}}(3\mu_{k-1}M_{k-1})^{2p_1\cdots p_{k-2}}\times$$
$$[\frac{1}{4}\frac{2p_1\cdots p_{k-2}}{2p_1\cdots p_{k-2}-1}]^{-(2p_1\cdots p_{k-2}-1)},$$

$$\phi_{k,2}(\cdot)\leqslant a\Theta\frac{1}{2q_1\cdots q_{k-2}}(3\mu_{k-1}M_{k-1})^{2q_1\cdots q_{k-2}}\times$$
$$[\frac{1}{4}\frac{2q_1\cdots q_{k-2}}{2q_1\cdots q_{k-2}-1}]^{-(2q_1\cdots q_{k-2}-1)}.$$

从而式(9)成立.

A.2. 存在光滑函数$\gamma_{k,2}(\cdot)\geqslant 0$, 使得式(10)成立.

证 由x_l^*,ξ_l的定义有, $|x_l|=y_l^{-1}(|\xi_l+y_l(x_l^*)|)$. 由引理4及$y_l^{-1}(\cdot)$的单调性容易得到, 当$2\leqslant l\leqslant k$时,

$$\begin{aligned}|x_l|\leqslant&\ y_l^{-1}(|\xi_l|)+y_l^{-1}[|y_l(x_l^*)|]=\\&y_l^{-1}(|\xi_l|)+y_l^{-1}(|\xi_{l-1}\beta_{l-1}(\cdot)|)\leqslant\\&y_l^{-1}(|\xi_l|)+(1+\beta_{l-1}(\cdot))y_l^{-1}(|\xi_{l-1}|),\end{aligned}$$

注意到$|x_1|=y_1^{-1}(|\xi_1|)$和$\beta_i(\cdot)$的光滑性, 由引理8有, 存在光滑函数$\hat\rho_k(x_1,\cdots,x_k)>0$, 使得

$$\sum_{l=1}^{k}|x_l|\leqslant\sum_{l=1}^{k}y_k^{-1}(|\xi_l|)\,\hat\rho_k(x_1,\cdots,x_k).\tag{A1}$$

故由引理3及$\Theta_1\leqslant\Theta_2$知

$$\begin{aligned}|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}f_k(\cdot)|\leqslant&\ \frac{d_k(\cdot)}{2}|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}x_{k+1}^{p_k}|+\\&\sum_{l=1}^{k}|\frac{\xi_k^2}{y_k^{-1}(\xi_k)}y_k^{-1}(\xi_l)|\bar f_k(\cdot)\hat\rho_k(\cdot)\Theta_2.\end{aligned}$$

利用A.1.的证明方法, 由

$$\Theta\geqslant\Theta_2^{\frac{2p_1\cdots p_{k-1}}{2p_1\cdots p_{k-1}-1}}a^{\frac{2-2p_1\cdots p_{k-1}}{2p_1\cdots p_{k-1}-1}},\Theta_2^{\frac{2q_1\cdots q_{k-1}}{2q_1\cdots q_{k-1}-1}}a^{\frac{2-2q_1\cdots q_{k-1}}{2q_1\cdots q_{k-1}-1}},$$

容易得到不等式(10)成立.

A.3. 存在光滑函数$\gamma_{k,3}(\cdot)\geqslant 0$, 使得式(11)成立.

证 由引理3和式(A1)知

$$\begin{aligned}|\dot x_l|\leqslant&\ \frac{3}{2}b_l\mu_l(\cdot)|x_{l+1}^{p_l}+x_{l+1}^{q_l}|+\\&\sum_{j=1}^{l}y_l^{-1}(|\xi_j|)\bar f_l(\cdot)\hat\rho_l(\cdot)\Theta_1,\end{aligned}$$

其中

$$\begin{aligned}&|x_{l+1}^{p_l}+x_{l+1}^{q_l}|=|y_l^{-1}[y_{l+1}(x_{l+1})]|\leqslant\\&y_l^{-1}[|\xi_{l+1}|+|y_{l+1}(x_{l+1}^*)|]\leqslant\\&y_l^{-1}(|\xi_{l+1}|)+y_l^{-1}(|y_{l+1}(x_{l+1}^*)|)=\\&y_l^{-1}(|\xi_{l+1}|)+y_l^{-1}(|\xi_l\beta_l(\cdot)|)\leqslant\\&y_l^{-1}(|\xi_{l+1}|)+y_l^{-1}(|\xi_l|(1+\beta_l(\cdot))).\end{aligned}$$

因此, 存在光滑函数$C_l(x_1,\cdots,x_l,\hat\Theta)>0$, 使得

$$|\dot x_l|\leqslant\sum_{j=1}^{l+1}y_l^{-1}(|\xi_j|)\,C_l(\cdot)\Theta_2.\tag{A2}$$

由引理7和$\dfrac{x}{y_k^{-1}(x)}$的单调性,

$$\begin{aligned}|\frac{\partial W_k}{\partial x_l}|\leqslant&\ |2-\frac{1}{p_1\cdots p_{k-1}}||\frac{\partial y_k(x_k^*)}{\partial x_l}|\times\\&|\frac{y_k(x_k)-y_k(x_k^*)}{y_k^{-1}(y_k(x_k)-y_k(x_k^*))}||x_k-x_k^*|.\end{aligned}$$

注意到$\xi_k=y_k(x_k)-y_k(x_k^*)$和

$$|x_k-x_k^*|=|y_k^{-1}(y_k(x_k))-y_k^{-1}(y_k(x_k^*))|\leqslant 2y_k^{-1}(|\xi_k|),$$

故

$$|\frac{\partial W_k}{\partial x_l}|\leqslant 4|\frac{\partial y_k(x_k^*)}{\partial x_l}||\xi_k|.\tag{A3}$$

对于$|\dfrac{\partial y_k(x_k^*)}{\partial x_l}|$, 存在光滑函数$\varphi_{k,l}(x_1,\cdots,x_k,\hat\Theta)\geqslant 0$, 使得

$$|\frac{\partial y_k(x_k^*)}{\partial x_l}|\leqslant\sum_{i=1}^{k-1}\frac{|\xi_i|}{|y_l^{-1}(\xi_i)|}\varphi_{k,l}(\cdot).\tag{A4}$$

这是因为, 当$k=2$时, $y_2(x_2^*)=-(x_1\beta_1(\cdot))$, 故

$$|\frac{\partial y_2(x_2^*)}{\partial x_1}|=|\beta_1(x_1)|+|x_1\frac{\partial\beta_1(x_1)}{\partial x_1}|,$$

由$\beta_1(\cdot)$的光滑性有式(A4)成立. 假设在$k-1$时, 存在光滑函数$\varphi_{k-1,l}(\cdot)\geqslant 0$, 使得

$$|\frac{\partial y_{k-1}(x_{k-1}^*)}{\partial x_l}|\leqslant\sum_{i=1}^{k-2}\frac{|\xi_i|}{|y_l^{-1}(\xi_i)|}\varphi_{k-1,l}(\cdot),\tag{A5}$$

则在k时,

$$\begin{aligned}|\frac{\partial y_k(x_k^*)}{\partial x_l}|\leqslant&\ |\frac{\partial\xi_{k-1}}{\partial x_l}|\beta_{k-1}(\cdot)+|\xi_{k-1}\frac{\partial\beta_{k-1}(\cdot)}{\partial x_l}|\leqslant\\&[|\frac{\partial y_{k-1}(x_{k-1})}{\partial x_l}|+|\frac{\partial y_{k-1}(x_{k-1}^*)}{\partial x_l}|]\beta_{k-1}(\cdot)+\\&|\xi_{k-1}\frac{\partial\beta_{k-1}(\cdot)}{\partial x_l}|.\end{aligned}$$

当$l\leqslant k-2$时, 注意到$|\dfrac{\partial y_{k-1}(x_{k-1})}{\partial x_l}|=0$, 由式(A5)

$$\begin{aligned}&|\frac{\partial y_k(x_k^*)}{\partial x_l}|\leqslant\\&|\xi_{k-1}\frac{\partial\beta_{k-1}(\cdot)}{\partial x_l}|+|\frac{\partial y_{k-1}(x_{k-1}^*)}{\partial x_l}|\beta_{k-1}(\cdot)\leqslant\\&|\frac{\xi_{k-1}}{y_l^{-1}(\xi_{k-1})}||y_l^{-1}(\xi_{k-1})\frac{\partial\beta_{k-1}(\cdot)}{\partial x_l}|+\\&\sum_{i=1}^{k-2}\frac{|\xi_i|}{|y_l^{-1}(\xi_i)|}\varphi_{k-1,l}(\cdot)\beta_{k-1}(\cdot),\end{aligned}$$

故存在光滑函数$\varphi_{k,l}(\cdot)\geqslant 0$使得式(A5)在$l\leqslant k-2$时成立.

当$l=k-1$时, $|\dfrac{\partial y_{k-1}(x_{k-1}^*)}{\partial x_{k-1}}|=0$, 故

$$|\frac{\partial y_k(x_k^*)}{\partial x_{k-1}}|\leqslant|\xi_{k-1}\frac{\partial\beta_{k-1}(\cdot)}{\partial x_{k-1}}|+|\frac{\partial y_{k-1}(x_{k-1})}{\partial x_{k-1}}|\beta_{k-1}(\cdot).$$

由引理7,

$$\begin{aligned}&\frac{1}{p_1\cdots p_{k-1}}|\frac{\partial y_{k-1}(x_{k-1})}{\partial x_{k-1}}|\leqslant\\&|\frac{\xi_{k-1}+y_{k-1}(x_{k-1}^*)}{y_{k-1}^{-1}(\xi_{k-1}+y_{k-1}(x_{k-1}^*))}|\leqslant\\&\frac{|\xi_{k-1}|}{y_{k-1}^{-1}(|\xi_{k-1}|)}+\frac{|y_{k-1}(x_{k-1}^*)|}{y_{k-1}^{-1}(|y_{k-1}(x_{k-1}^*)|)}.\end{aligned}$$

注意到$\dfrac{x}{y_i^{-1}(x)}$和$y_i^{-1}(x)$的单调性有

$$\begin{aligned}&\frac{|y_{k-1}(x_{k-1}^*)|}{y_{k-1}^{-1}(|y_{k-1}(x_{k-1}^*)|)}=\frac{|\xi_{k-2}\beta_{k-2}(\cdot)|}{y_{k-1}^{-1}(|\xi_{k-2}\beta_{k-2}(\cdot)|)}\leqslant\\&\frac{|\xi_{k-2}(1+\beta_{k-2}(\cdot))|}{y_{k-1}^{-1}(|\xi_{k-2}(1+\beta_{k-2}(\cdot))|)}\leqslant\frac{|\xi_{k-2}(1+\beta_{k-2}(\cdot))|}{y_{k-1}^{-1}(|\xi_{k-2}|)}=\\&\frac{|\xi_{k-2}|}{y_{k-1}^{-1}(|\xi_{k-2}|)}(1+\beta_{k-2}(\cdot)),\end{aligned}$$

故

$$\begin{aligned}&|\frac{\partial y_k(x_k^*)}{\partial x_{k-1}}|\leqslant\\&|\xi_{k-1}\frac{\partial\beta_{k-1}(\cdot)}{\partial x_{k-1}}|+(\frac{|\xi_{k-1}|}{y_{k-1}^{-1}(|\xi_{k-1}|)}+\\&\frac{|\xi_{k-2}|}{y_{k-1}^{-1}(|\xi_{k-2}|)}(1+\beta_{k-2}(\cdot)))p_1\cdots p_{k-1}\beta_{k-1}(\cdot),\end{aligned}$$

从而存在光滑函数$\varphi_{k,k-1}(\cdot) \geqslant 0$使得式(A4)在$l = k-1$时成立.

这样, 由式(A2)–(A4)有

$$\sum_{l=1}^{k-1} |\frac{\partial W_k(\cdot)}{\partial x_l}\dot{x}_l| \leqslant \sum_{l=1}^{k-1}[4(\sum_{i=1}^{k-1}\frac{|\xi_i|}{|y_l^{-1}(\xi_i)|}\varphi_{k,l}(\cdot)) \times |\xi_k|(\sum_{j=1}^{l+1} y_l^{-1}(|\xi_j|)C_l(\cdot)\Theta_2)].$$

利用A.1.的证明方法, 由Θ的定义, 容易得到式(11)成立.

A.4. 存在光滑函数$\gamma_{k,4}(\cdot) \geqslant 0$, 使得式(15)成立.

证 与式(A3)相同, 容易得到

$$|\frac{\partial W_i}{\partial \hat{\Theta}}| \leqslant 4|\frac{\partial y_i(x_i^*)}{\partial \hat{\Theta}}||\xi_i|,$$

故由引理2

$$\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial \hat{\Theta}}\tau_k(\cdot)\xi_k^2 \leqslant \sum_{i=2}^{k-1} 4|\frac{\partial y_i(x_i^*)}{\partial \hat{\Theta}}||\xi_i|\tau_k(\cdot)\xi_k^2 \leqslant \sum_{i=2}^{k-1}\frac{1}{8a}\xi_i^2 + 32a(\frac{\partial y_i(x_i^*)}{\partial \hat{\Theta}}\tau_k(\cdot)\xi_k^2)^2;$$

同理, 由

$$\rho_k(x_1, \cdots, x_k, \hat{\Theta}) = \sum_{i=1}^{k}\tau_i(\cdot)\xi_i^2,$$

有

$$\frac{\partial W_k}{\partial \hat{\Theta}}\rho_k(\cdot) \leqslant \sum_{i=1}^{k} 4|\frac{\partial y_k(x_k^*)}{\partial \hat{\Theta}}||\xi_k|\tau_i(\cdot)\xi_i^2 \leqslant \sum_{i=1}^{k}\frac{1}{8a}\xi_i^2 + 32a(\frac{\partial y_k(x_k^*)}{\partial \hat{\Theta}}\tau_k(\cdot)\xi_k\xi_i)^2.$$

从而有存在光滑函数$\gamma_{k,4}(\cdot) \geqslant 0$, 使得式(15)成立.

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