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在离散观测和反馈延迟下的混杂随机系统镇定

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摘要: 对混杂随机系统的状态反馈控制近年来引起了广泛的关注. 一个更现实也更经济的情况是对状态的观测不是连续时间而是离散时间的. 同时, 现实中绝大多数的观测和反馈系统都或多或少会存在时滞现象. 因此, 讨论这种基于离散时间观测同时带有观测反馈时滞的混杂随机系统的反馈控制是很有意义的. 特别地, 通过使用一个李雅普诺夫泛函, 不仅可以得到H无穷稳定、渐近稳定和指数稳定, 而且还能显著地改善对时滞上界的要求. 本文是文献[20]工作的深入和推广.

关键词: 离散时间观测; 反馈延迟; 李雅普诺夫泛函; 随机控制系统; H无穷稳定; 渐近稳定; 指数稳定

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Stabilisation of hybrid stochastic systems under discrete observation and sample delay

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Abstract: State feedback controls for hybrid stochastic differential equations have attracted lots of attention in recent years. It is more economic and practical that the states are observed at discrete time instead of continuous time. In addition, most observations and feedback systems have some time delays in practice. Therefore, it is interesting to investigate feedback controls based on discrete-time observations with sample delays for hybrid systems. In this paper, exponential stabilisations both in H_∞ and asymptotical sense are discussed using Lyapunov functionals. The upper bound of the delay time is also improved. This work is devoted as a continuous research to Ref. [20].

Key words: discrete-time state observation; sample delay; Lyapunov functional; stochastic control system; H_∞ stability; asymptotically stability; exponential stability

1 Introduction

Hybrid stochastic differential equations (SDEs) have been attracting a lot of attention in recent years. We refer the readers to monographs [1–2] for the detailed introduction. As an important aspect of the study on hybrid SDEs, the automatic control with the emphasis on the asymptotic stability analysis has been broadly discussed. We just mention some of the works [3–18] and the references therein. One classical problem of this field is to design a control function $u(x(t))$ embedded into the drift coefficient such that the modified system

$$\mathrm{d}x(t) = [f(x(t), r(t), t) + u(x(t), r(t), t)]\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}B(t) \tag{1}$$

is stable, while the original system, (1) without $u(x(t), r(t), t)$, is unstable.

Due to various reasons and unexpected effects, the feedback control based on state observation suffers a time delay τ_0 in practice. To tackle this drawback, Mao in [19] analysed the asymptotic stability of the following model

$$\mathrm{d}x(t) = [f(x(t), r(t), t) + u(x(t-\tau_0), r(t), t)]\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}B(t). \tag{2}$$

On the other hand, it is expensive and impractical to design the control function $u(x(t), r(t), t)$ based on the continuous state $x(t)$. Actually, in practice the state can only be observed at discrete time point. In [20], the author initialised this idea for hybrid SDEs and developed the technique of feedback controls based on discrete-time state observation. We need to mention that for the deterministic counterparts this idea has been discussed (see for examples [21–25]).

In [20], the author introduced the theory on stabili-

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sation that the feedback function can be designed based on a sequence of countable states $x(\lfloor t/\tau \rfloor \tau)$, where τ is a positive number and $\lfloor t/\tau \rfloor$ denotes the maximal integer not large than t/τ. Then the control function is constructed as $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$. One may see that only the states at $0, \tau, 2\tau, \cdots$ are required. Therefore, the controlled system becomes

$$\mathrm{d}x(t) = [f(x(t), r(t), t) + u(x(\lfloor t/\tau \rfloor \tau), r(t), t)]\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}B(t). \tag{3}$$

Due to the general techniques used in proofs in [20], the restriction on τ was quite strong in that paper. To tackle it, Mao and his group modified the techniques and released the restriction on τ in [26–27].

The time-delayed feedback control has been used in a large variety of systems in physics, chemistry, biology, medicine, and engineering [28–32]. In this paper, we consider a more realistic and economic controlled hybrid system. We observe that due to some practical problems the state at a discrete time $k\tau$ may be received after a time delay. Alternatively, the state feedback control signal may return the original system with a time delay. In both cases, we denote the delay time by a constant τ_0. Based on this finding, we construct the feedback control based on discrete-time state observations with a time delay in this paper. The new controlled system reads as (please find the details in Section 2)

$$\mathrm{d}x(t) = [f(x(t), r(t), t) + u(x(\delta(t)), r(t), t)]\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}B(t), \tag{4}$$

where $\delta(t) = \lfloor t/\tau \rfloor \tau - \tau_0$, τ_0 is the length of delaying time and τ is the discrete time gap.

One may notice that (3) is already a stochastic delay differential equation because of the control function $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ having time delay when $k\tau < t < (k+1)\tau$ for any positive integer k. But there is no delay when $t = k\tau$, which means the state observation of $x(k\tau)$ returns immediately. However, the realistic case is that the time delay always exists in the control function $u(x(k\tau - \tau_0))$ even at $t = k\tau$. Thus some new difficulties arise in the proof of this work compared with [20, 26–27].

This paper is constructed in the following way. In Section 2, some mathematical preliminaries are given. Section 3 sees the application of the Lyapunov functional to the study on the H_∞ stability and the mean square asymptotical stability. Section 4 is devoted to the analysis of the stability rate. Examples and numerical simulations are displayed in Section 5. Section 6 concludes this paper with some possible future research.

2 Mathematical preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions that it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $B(t) = (B_1(t), \cdots, B_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion defined on the probability space. For a vector or matrix A, A^{T} denotes its transpose. For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$ and $\|A\| = \max\{|Ax| : |x| = 1\}$ denote the trace and operator norms of a matrix A, respectively. For a symmetric matrix A, i.e. $A = A^{\mathrm{T}}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote its smallest and largest eigenvalues, respectively. We define A as non-positive and negative definite by $A \leqslant 0$ and $A < 0$, respectively. Denote by $L^2_{\mathcal{F}_t}(\mathbb{R}^n)$, the family of all $\mathcal{F}_t$-measurable $\mathbb{R}^n$-valued random variables ξ such that $\mathrm{E}|\xi|^2 < \infty$, where E is the expectation with respect to the probability measure $\mathbb{P}$. For a non-negative real number a, let $\lfloor a \rfloor$ denote the integer part of a and $\lceil a \rceil$ denote the smallest integer not less than a. If $a, b \in \mathbb{R}$, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. I_A denotes the indicator function, which means $I_\omega = 1$ for $\omega \in A$ and $I_\omega = 0$ otherwise.

Let $r(t)$ denotes a right-continuous Markov chain which taking values in a finite state space $S = \{1, 2, \cdots, N\}$. Set its generator by $\Gamma = (\gamma_{ij})_{N\times N}$, and

$$\mathrm{P}\{r(t+\varDelta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\varDelta + \mathrm{o}(\varDelta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\varDelta + \mathrm{o}(\varDelta), & \text{if } i = j. \end{cases}$$

Here $\gamma_{ij} \geqslant 0$ means the transition rate from i to j. And $\gamma_{ii} = -\sum\limits_{j\neq i}\gamma_{ij}$. As usual, the Markov chain $r(\cdot)$ and the Brownian motion $w(\cdot)$ are independent. It is known that $r(t)$ is a time-continuous and state-discrete Markov chain. Thus, for any finite subinterval of t when $t \in [0, \infty)$, $r(t)$ only have a finite number of jumps. Except these jumps, almost all paths of $r(t)$ are constant. We emphasise that almost all sample paths of $r(t)$ are right continuous.

The hybrid system that we will investigate is an n-dimensional unstable hybrid system

$$\mathrm{d}y(t) = f(y(t), r(t), t)\mathrm{d}t + g(y(t), r(t), t)\mathrm{d}B(t), \tag{5}$$

for $t \geqslant -\tau_0$. Its initial data is $y(-\tau_0) = y_0 \in L^2_{\mathcal{F}_0}(\mathbb{R}^n)$ and $r(-\tau_0) = r_0$.

Now for a constant time delay $\tau_0 > 0$, one could design the control function based on the discrete-time observations with a fixed gap $\tau > 0$. Then the controlled system is a hybrid SDEs as follows

$$\mathrm{d}x(t) = [f(x(t), r(t), t) + u(x(\delta(t)), r(t), t)]\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}B(t), \tag{6}$$

for $t \geqslant 0$, where

$$\delta(t) = \lfloor \frac{t}{\tau} \rfloor \tau - \tau_0. \tag{7}$$

It should be noticed that original system (5) only provides initial data at $t = -\tau_0$, but the controlled system (6) requires a segment of initial data $x_\theta := \{x(\theta),$

$-\tau_0 \leqslant \theta \leqslant 0\}$. To fill up this gap, we let the original system (5) to evolve for a period of $t \in [-\tau_0, 0]$. During this period we would observe the states of systems at the discrete time point. Then these observations are regarded as the initial data for the controlled system (6). Therefore, we have that

$$y(t) = x(t), \text{ for } -\tau_0 \leqslant t \leqslant 0. \tag{8}$$

We define a positive constant h here, as it will be used through the rest of the paper.

$$h = \lceil \frac{\tau_0}{\tau} \rceil. \tag{9}$$

The following assumptions are imposed.

Assumption 2.1 Assume that the functions

$$f : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$$

satisfy the local Lipschitz condition (see e.g. [1, 33–35]). Besides, for all $(x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}_+$, they satisfy

$$|f(x, i, t)| \leqslant K_1|x|, \ |g(x, i, t)| \leqslant K_3|x|, \tag{10}$$

where K_1 and K_3 are positive constants.

It is easy to see from (10) that

$$f(0, i, t) = 0, \ g(0, i, t) = 0, \tag{11}$$

for all $(i, t) \in S \times \mathbb{R}_+$.

Assumption 2.2 Assume that the controlling function $u : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n$ is globally Lipschitz continuous that

$$|u(x, i, t) - u(y, i, t)| \leqslant K_2|x - y|, \tag{12}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}_+$, where K_2 is a positive constant. In addition, it satisfies

$$u(0, i, t) = 0 \tag{13}$$

for all $(i, t) \in S \times \mathbb{R}_+$, which indicates

$$|u(x, i, t)| \leqslant K_2|x|. \tag{14}$$

It should be pointed out that from [1] we know that the original system (5) does not explode during $t \in [-\tau_0, 0]$ under Assumption 2.1, that is there exists a positive constant C_0 such that

$$\sup_{-\tau_0\leqslant t\leqslant 0} \mathrm{E}|x(t)|^2 = \sup_{-\tau_0\leqslant t\leqslant 0} \mathrm{E}|y(t)|^2 \leqslant C_0. \tag{15}$$

In the proofs of theorems in the next two sections, some additional data $x_\theta = \{x(\theta), -2(\tau_0 + \tau) \leqslant \theta < -\tau_0\}$ are required. However, the actual values of these data are of no effect on our results. Bearing this in mind, we simply set

$$x(\theta) = y(-\tau_0), \text{ for } \theta \in [-2(\tau_0 + \tau), -\tau_0). \tag{16}$$

It needs to be pointed out that $x_\theta = \{x(\theta), -2(\tau_0 + \tau) \leqslant \theta < -\tau_0\}$ are only required for the theoretical analysis but not in practice.

3 Asymptotic stabilization

We design a Lyapunov functional as follows

$$\begin{aligned} &V(\hat{x}_t, \hat{r}_t, t) = U(x(t), r(t), t)+ \\ &\theta\int_{t-\tau-\tau_0}^{t}\int_{s}^{t}[(\tau + \tau_0)|f(x(v), r(v), v)+ \\ &u(x(\delta_v), r(v), v)I_{v\geqslant 0}|^2 + |g(x(v), r(v), v)|^2]\mathrm{d}v\mathrm{d}s, \end{aligned} \tag{17}$$

for $t \geqslant 0$, where $\theta > 0$ is determined later on. We let $\delta_v := \delta(v)$. One may see that the Lyapunov functional is based on the segments $\hat{x}_t := \{x(t + s) : -2(\tau + \tau_0) \leqslant s \leqslant 0\}$ and $\hat{r}_t := \{r(t + s) : -2(\tau + \tau_0) \leqslant s \leqslant 0\}$ for $t \geqslant 0$. $\hat{x}_0$ is defined in Section 2 by (8) and (16), and $\hat{r}_0$ is defined in the similar way. Let $U \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of all non-negative functions which are continuously twice differentiable in x and once in t. As usual, let us define an operator $\mathcal{L}U$ from $\mathbb{R}^n \times S \times \mathbb{R}_+$ to $\mathbb{R}$ by

$$\begin{aligned} &\mathcal{L}U(x, i, t) = \\ &U_t(x, i, t) + U_{\mathrm{x}}(x, i, t)[f(x, i, t) + u(x, i, t)] + \\ &\frac{1}{2}\mathrm{tr}[g^{\mathrm{T}}(x, i, t)U_{\mathrm{xx}}(x, i, t)g(x, i, t)] + \\ &\sum_{j=1}^{N}\gamma_{ij}U(x, j, t), \end{aligned} \tag{18}$$

where

$$U_t(x, i, t) = \frac{\partial U(x, i, t)}{\partial t},$$

$$U_{\mathrm{x}}(x, i, t) = (\frac{\partial U(x, i, t)}{\partial x_1}, \cdots, \frac{\partial U(x, i, t)}{\partial x_n}),$$

and

$$U_{\mathrm{xx}}(x, i, t) = (\frac{\partial^2 U(x, i, t)}{\partial x_i \partial x_j})_{n\times n}.$$

Now we can present our first stability result in the sense of H_∞.

Theorem 3.1 For any system (6) that satisfies Assumptions 2.1 and 2.2, if there exist functions $U \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}_+; \mathbb{R}_+)$ and positive numbers λ_1, λ_2 so that

$$\mathcal{L}U(x, i, t) + \lambda_1|U_{\mathrm{x}}(x, i, t)|^2 \leqslant -\lambda_2|x|^2, \tag{19}$$

for any $(x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}_+$. Then

$$\int_0^\infty \mathrm{E}|x(s)|^2\mathrm{d}s < \infty, \tag{20}$$

for all initial data $x_\theta := \{x(\theta), -\tau_0 \leqslant \theta \leqslant 0\}$ and $r_\theta := \{r(\theta), -\tau_0 \leqslant \theta \leqslant 0\}$. If $\tau > 0$ and $\tau_0 > 0$ fulfill

$$\lambda_2 > \frac{(\tau + \tau_0)K_2^2}{\lambda_1}[2(\tau + \tau_0)(K_1^2 + 2K_2^2) + K_3^2], \tag{21}$$

and

$$\tau + \tau_0 \leqslant \frac{1}{4K_2}, \tag{22}$$

then the system (6) is H_∞-stable.

Proof From the generalized Itô formula (see e.g.

[1,6]), one can see that

$$\mathrm{d}V(\hat{x}_t,\hat{r}_t,t)=LV(\hat{x}_t,\hat{r}_t,t)\mathrm{d}t+\mathrm{d}M(t),\quad (23)$$

for $t\geqslant 0$. Here $M(t)$ is a continuous martingale whose explicit form is of no use, and $M(0)=0$. By Assumptions 2.1, 2.2 and (19), it is easy to get

$$\begin{aligned}&LV(\hat{x}_t,\hat{r}_t,t)\leqslant\\&\mathcal{L}U(x(t),r(t),t)+\lambda_1|U_x(x(t),r(t),t)|^2+\\&\theta(\tau+\tau_0)[2(\tau+\tau_0)(K_1^2+2K_2^2)+K_3^2]|x(t)|^2+\\&(\frac{K_2^2}{4\lambda_1}+4\theta(\tau+\tau_0)^2K_2^2)|x(t)-x(\delta_t)|^2-\\&\theta\int_{t-\tau-\tau_0}^{t}[(\tau+\tau_0)|f(x(s),r(s),s)+\\&u(x(\delta_s),r(s),s)I_{s\geqslant 0}|^2+|g(x(s),r(s),s)|^2]\mathrm{d}s\leqslant\\&-\lambda|x(t)|^2+(\frac{K_2^2}{4\lambda_1}+4\theta(\tau+\tau_0)^2K_2^2)|x(t)-\\&x(\delta_t)|^2-\theta\int_{t-\tau-\tau_0}^{t}[(\tau+\tau_0)|f(x(s),r(s),s)+\\&u(x(\delta_s),r(s),s)I_{s\geqslant 0}|^2+|g(x(s),r(s),s)|^2]\mathrm{d}s,\end{aligned}\quad (24)$$

where

$$\begin{aligned}&\lambda=\lambda(\theta,\tau,\tau_0):=\\&\lambda_2-\theta(\tau+\tau_0)[2(\tau+\tau_0)(K_1^2+2K_2^2)+K_3^2].\end{aligned}\quad (25)$$

Noting that $t-\delta_t\leqslant\tau+\tau_0$ and $u(x(\delta_t),r(t),t)=0$ when $t<h\tau$, we sperate the time interval into two parts. Firstly, for all $t\geqslant h\tau$, which means $\delta_t\geqslant 0$, it is not hard to get from (6) that

$$\begin{aligned}&\mathrm{E}|x(t)-x(\delta_t)|^2\leqslant\\&2\mathrm{E}\int_{\delta_t}^{t}[(\tau+\tau_0)|f(x(s),r(s),s)+\\&u(x(\delta_s),r(s),s)|^2+|g(x(s),r(s),s)|^2]\mathrm{d}s=\\&2\mathrm{E}\int_{\delta_t}^{t}[(\tau+\tau_0)|f(x(s),r(s),s)+\\&u(x(\delta_s),r(s),s)I_{s\geqslant 0}|^2+|g(x(s),r(s),s)|^2]\mathrm{d}s.\end{aligned}\quad (26)$$

Secondly, for all $0\leqslant t<h\tau$, we have

$$\begin{aligned}&x(t)-x(\delta_t)=\\&\int_0^t[f(x(s),r(s),s)+u(x(\delta_s),r(s),s)]\mathrm{d}s+\\&\int_0^t g(x(s),r(s),s)\mathrm{d}B(s)+\int_{\delta_t}^0 f(x(s),r(s),s)\mathrm{d}s+\\&\int_{\delta_t}^0 g(x(s),r(s),s)\mathrm{d}B(s)=\\&\int_{\delta_t}^t[f(x(s),r(s),s)+u(x(\delta_s),r(s),s)I_{s\geqslant 0}]\mathrm{d}s+\\&\int_{\delta_t}^t g(x(s),r(s),s)\mathrm{d}B(s),\end{aligned}\quad (27)$$

which then indicates that

$$\begin{aligned}&\mathrm{E}|x(t)-x(\delta_t)|^2\leqslant\\&2\mathrm{E}\int_{\delta_t}^{t}[(\tau+\tau_0)|f(x(s),r(s),s)+\\&u(x(\delta_s),r(s),s)I_{s\geqslant 0}|^2+|g(x(s),r(s),s)|^2]\mathrm{d}s\end{aligned}\quad (28)$$

holds for $0\leqslant t<h\tau$. Together with (26), we see that (28) holds for any $t\geqslant 0$. Now we choose

$$\theta=\frac{K_2^2}{\lambda_1}.\quad (29)$$

Together with (22)(29) yields

$$\mathrm{E}(LV(\hat{x}_t,\hat{r}_t,t))\leqslant-\lambda\mathrm{E}|x(t)|^2,\quad (30)$$

and it is easy to see that $\lambda>0$ from (21). Then by (23) we can get

$$0\leqslant\mathrm{E}(V(\hat{x}_t,\hat{r}_t,t))\leqslant C_1-\lambda\int_0^t\mathrm{E}|x(s)|^2\mathrm{d}s,\quad (31)$$

for $t\geqslant 0$, where C_1 is defined as

$$\begin{aligned}&C_1=V(\hat{x}_0,\hat{r}_0,0)=\\&U(x_0,r_0,0)+\theta\int_{-\tau-\tau_0}^{0}\int_s^0[(\tau+\tau_0)\\&|f(x(v),r(v),v)|^2+|g(x(v),r(v),v)|^2]\mathrm{d}v\mathrm{d}s.\end{aligned}\quad (32)$$

Then we can see that

$$\int_0^\infty\mathrm{E}|x(s)|^2\mathrm{d}s\leqslant C_1/\lambda.$$

The proof is completed.

As same as [27], we will state that $\lim\limits_{t\to\infty}\mathrm{E}(|x(t)|^2)=0$ as our second result. Let us introduce an useful Lemma which will be used in the next theorem and next section. The proof of this Lemma is in the Appendix.

Lemma 3.2 Let Assumptions 2.1 and 2.2 hold. If $\gamma\geqslant 0$, and $\tau,\tau_0>0$ satisfy that $2(h+1)\bar{K}(\tau,\tau_0)<1$, where

$$K(\tau,\tau_0)=\tau[6(\tau+\tau_0)K_1^2+6K_3^2+3(\tau+\tau_0)K_2^2]\cdot\mathrm{e}^{[6(\tau+\tau_0)K_1^2+6K_3^2](\tau+\tau_0)},\quad (33)$$

and

$$\bar{K}(\tau,\tau_0)=K(\tau,\tau_0)\mathrm{e}^{\gamma h\tau}.\quad (34)$$

Then the solution of system (6) satisfies

$$\begin{aligned}&\int_0^t\mathrm{e}^{\gamma s}\mathrm{E}|x(s)-x(\delta(s))|^2\mathrm{d}s\leqslant\\&\frac{2(h+1)\bar{K}(\tau,\tau_0)}{1-2(h+1)\bar{K}(\tau,\tau_0)}\int_0^t\mathrm{e}^{\gamma s}\mathrm{E}|x(s)|^2\mathrm{d}s+\\&\frac{\tau(h+h^2)\bar{K}(\tau,\tau_0)}{2-4(h+1)\bar{K}(\tau,\tau_0)}\sup_{\theta\in[-\tau_0,0]}\mathrm{E}|x(\theta)|^2,\end{aligned}\quad (35)$$

for any $t\geqslant 0$.

Now we state our second main result.

Theorem 3.3 The controlled system (6) is asymptotically stable in the mean square sense under the same conditions of Theorem 3.1. That is

$$\lim_{t\to\infty}\mathrm{E}|x(t)|^2=0,$$

for all initial data $x_\theta:=\{x(\theta),-\tau_0\leqslant\theta\leqslant 0\}$ and $r_\theta:=\{r(\theta),-\tau_0\leqslant\theta\leqslant 0\}$.

Proof As same as [27], by Assumptions 2.1–2.2

and Itô formula, one can get

$$\mathrm{E}|x(t)|^2 \leqslant |x_0|^2 + C\int_0^t \mathrm{E}|x(s)|^2\mathrm{d}s + C\int_0^t \mathrm{E}|x(s)-x(\delta_s)|^2\mathrm{d}s. \tag{36}$$

Here C denotes a positive constant, whose detailed form is of no use. Let $\gamma = 0$, by Lemma 3.2 we have

$$\int_0^t \mathrm{E}|x(s)-x(\delta(s))|^2\mathrm{d}s \leqslant \frac{2(h+1)K(\tau,\tau_0)}{1-2(h+1)K(\tau,\tau_0)}\int_0^t \mathrm{E}|x(s)|^2\mathrm{d}s + \frac{\tau(h+h^2)K(\tau,\tau_0)}{2-4(h+1)K(\tau,\tau_0)}\sup_{\theta\in[-\tau_0,0]}\mathrm{E}|x(\theta)|^2. \tag{37}$$

Together with (36), we obtain that

$$\mathrm{E}|x(t)|^2 \leqslant C\sup_{\theta\in[-\tau_0,0]}\mathrm{E}|x(\theta)|^2 + C\int_0^t \mathrm{E}|x(s)|^2\mathrm{d}s. \tag{38}$$

Theorem 3.1 guarantees that

$$\mathrm{E}|x(t)|^2 \leqslant C, \tag{39}$$

for all $t \geqslant 0$. Then by the same techniques used in [27], it is not hard to get that $\mathrm{E}|x(t)|^2$ is uniformly continuous in t on $\mathbb{R}_+$. Thus the desired assertion $\lim_{t\to\infty}\mathrm{E}|x(t)|^2 = 0$ holds naturally. The proof is completed.

4 Exponential stabilization

We have discussed the stabilisation in the H_∞ sense and asymptotically stable in the mean square sense. In practice, one may further ask for the stabilisation rate. This section is devoted for this part.

Theorem 4.1 Let Assumptions 2.1 and 2.2 hold for (6). If there exist two positive constants c_1 and c_2 such that the function $U(x,i,t)$ fulfills (19) and

$$c_1|x|^2 \leqslant U(x,i,t) \leqslant c_2|x|^2, \tag{40}$$

for all $(x,i,t) \in \mathbb{R}^n \times S \times \mathbb{R}_+$, then the controlled system (6) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathrm{E}|x(t)|^2) \leqslant -\gamma, \tag{41}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) \leqslant -\frac{\gamma}{2}\quad \text{a.s.}, \tag{42}$$

for all initial data $x_\theta := \{x(\theta), -\tau_0 \leqslant \theta \leqslant 0\}$ and $r_\theta := \{r(\theta), -\tau_0 \leqslant \theta \leqslant 0\}$. Here, $\tau > 0$ and $\tau_0 > 0$ satisfies (21)–(22) and $2(h+1)\bar{K}(\tau,\tau_0) < 1$. λ and θ are set to be the same as those in Theorem 3.1 by (25) and (29), and the rate $\gamma > 0$ is the unique root to the following equation

$$(\tau+\tau_0)\gamma\mathrm{e}^{(\tau+\tau_0)\gamma}\Big(H_1 + H_2\frac{2(h+1)K(\tau,\tau_0)\mathrm{e}^{h\tau\gamma}}{1-2(h+1)K(\tau,\tau_0)\mathrm{e}^{h\tau\gamma}}\Big) + \gamma c_2 = \lambda, \tag{43}$$

in which

$$\begin{cases} H_1 = \theta(\tau+\tau_0)(2(\tau+\tau_0)(K_1^2+2K_2^2)+K_3^2), \\ H_2 = 4\theta(\tau+\tau_0)^2K_2^2. \end{cases} \tag{44}$$

Proof From the generalized Itô formula, we see that

$$c_1\mathrm{e}^{\gamma t}\mathrm{E}|x(t)|^2 \leqslant C_1 + \int_0^t \mathrm{e}^{\gamma z}[\gamma\mathrm{E}(V(\hat{x}_z,\hat{r}_z,z)) - \lambda\mathrm{E}|x(z)|^2]\mathrm{d}z. \tag{45}$$

By the Lyapunov functional (17), we have

$$\mathrm{E}(V(\hat{x}_z,\hat{r}_z,z)) \leqslant c_2\mathrm{E}|x(z)|^2 + \mathrm{E}(\bar{V}(\hat{x}_z,\hat{r}_z,z)), \tag{46}$$

where

$$\bar{V}(\hat{x}_t,\hat{r}_t,t) := \theta\int_{t-\tau-\tau_0}^t\int_s^t [(\tau+\tau_0)|f(x(v),r(v),v) + u(x(\delta_v),r(v),v)I_{v\geqslant 0}|^2 + |g(x(v),r(v),v)|^2]\mathrm{d}v\mathrm{d}s. \tag{47}$$

Using Assumptions 2.1 and 2.2, we obtain that

$$\begin{aligned}&\mathrm{E}(\bar{V}(\hat{x}_z,\hat{r}_z,z)) \leqslant \\ &\theta(\tau+\tau_0)\int_{z-\tau-\tau_0}^z [(2(\tau+\tau_0)K_1^2+K_3^2)\mathrm{E}|x(v)|^2 + 2(\tau+\tau_0)K_2^2\mathrm{E}|x(\delta_v)|^2I_{v\geqslant 0}]\mathrm{d}v \leqslant \\ &\theta(\tau+\tau_0)\int_{z-\tau-\tau_0}^z [(2(\tau+\tau_0)(K_1^2+2K_2^2)+K_3^2)\mathrm{E}|x(v)|^2 + 4(\tau+\tau_0)K_2^2\mathrm{E}|x(v)-x(\delta_v)|^2I_{v\geqslant 0}]\mathrm{d}v \leqslant \\ &\int_{z-\tau-\tau_0}^z [H_1\mathrm{E}|x(v)|^2 + H_2\mathrm{E}|x(v)-x(\delta_v)|^2I_{v\geqslant 0}]\mathrm{d}v.\end{aligned} \tag{48}$$

Together with (45), one can get that, for $t \geqslant \tau+\tau_0$,

$$c_1\mathrm{e}^{\gamma t}\mathrm{E}|x(t)|^2 \leqslant C - (\lambda-\gamma c_2)\int_{\tau+\tau_0}^t \mathrm{e}^{\gamma z}\mathrm{E}|x(z)|^2\mathrm{d}z + \int_{\tau+\tau_0}^t \mathrm{e}^{\gamma z}\int_{z-\tau-\tau_0}^z [\gamma H_1\mathrm{E}|x(v)|^2 + \gamma H_2\mathrm{E}|x(v)-x(\delta_v)|^2I_{v\geqslant 0}]\mathrm{d}v\mathrm{d}z. \tag{49}$$

Here C is a positive constant whose accurate value is of no use. Now we have

$$\int_{\tau+\tau_0}^t \mathrm{e}^{\gamma z}\Big(\int_{z-\tau-\tau_0}^z \mathrm{E}|x(v)|^2\mathrm{d}v\Big)\mathrm{d}z \leqslant \int_0^t \mathrm{E}|x(v)|^2\Big(\int_v^{v+\tau+\tau_0}\mathrm{e}^{\gamma z}\mathrm{d}z\Big)\mathrm{d}v \leqslant (\tau+\tau_0)\mathrm{e}^{(\tau+\tau_0)\gamma}\int_0^t \mathrm{e}^{\gamma v}\mathrm{E}|x(v)|^2\mathrm{d}v, \tag{50}$$

and by Lemma 3.2 we see that

$$\int_{\tau+\tau_0}^t \mathrm{e}^{\gamma z}\Big(\int_{z-\tau-\tau_0}^z \mathrm{E}|x(v)-x(\delta_v)|^2I_{v\geqslant 0}\mathrm{d}v\Big)\mathrm{d}z \leqslant (\tau+\tau_0)\mathrm{e}^{(\tau+\tau_0)\gamma}\int_0^t \mathrm{e}^{\gamma v}\mathrm{E}|x(v)-x(\delta_v)|^2\mathrm{d}v \leqslant$$

$$(\tau+\tau_0)\mathrm{e}^{(\tau+\tau_0)\gamma}\frac{2(h+1)\bar{K}(\tau,\tau_0)}{1-2(h+1)\bar{K}(\tau,\tau_0)}\times$$
$$\int_0^t \mathrm{e}^{\gamma s}\mathrm{E}|x(s)|^2\mathrm{d}s+(\tau+\tau_0)\mathrm{e}^{(\tau+\tau_0)\gamma}\times$$
$$\frac{\tau(h+h^2)\bar{K}(\tau,\tau_0)}{2-4(h+1)\bar{K}(\tau,\tau_0)}\sup_{\theta\in[-\tau_0,0]}\mathrm{E}|x(\theta)|^2. \tag{51}$$

Together with (50), we see from (49) that

$$c_1\mathrm{e}^{\gamma t}\mathrm{E}|x(t)|^2\leqslant$$
$$C+[(\tau+\tau_0)\gamma\mathrm{e}^{(\tau+\tau_0)\gamma}(H_1+$$
$$H_2\frac{2(h+1)K(\tau,\tau_0)\mathrm{e}^{h\tau\gamma}}{1-2(h+1)K(\tau,\tau_0)\mathrm{e}^{h\tau\gamma}})+\gamma c_2-\lambda]$$
$$\int_0^t \mathrm{e}^{\gamma z}\mathrm{E}|x(z)|^2\mathrm{d}z. \tag{52}$$

Therefore, by using (43) in Theorem 3.1 we can achieve that

$$c_1\mathrm{e}^{\gamma t}\mathrm{E}|x(t)|^2\leqslant C,\ \ \forall t\geqslant\tau+\tau_0. \tag{53}$$

The desired assertion (41) and (42) follows.

Theorem 4.1 shows that the exponential stabilisation result in the general form. One may ask how to design the Lyapunov function $U(x,i,t)$. Thus, we give the next corollary. Set $U(x,i,t)=x^{\mathrm{T}}Q_ix$, where Q_i's are all symmetric positive-definite $n\times n$ matrices.

Corollary 4.2 For any system (6) that satisfied Assumptions 2.1 and 2.2. If there exist symmetric positive-definite matrices $Q_i\in\mathbb{R}^{n\times n}$ $(i\in S)$ so that

$$2x^{\mathrm{T}}Q_i[f(x,i,t)+u(x,i,t)]+$$
$$\mathrm{trace}[g^{\mathrm{T}}(x,i,t)Q_i(x,i,t)g(x,i,t)]+$$
$$\sum_{j=1}^{N}\gamma_{ij}x^{\mathrm{T}}Q_jx\leqslant-\lambda_3|x|^2, \tag{54}$$

for all $(x,i,t)\in\mathbb{R}^n\times S\times\mathbb{R}_+$, where λ_3 is a positive constant. Then the assertions of Theorem 4.1 hold if $\tau>0$ and $\tau_0>0$ satisfied (21)–(22) and $2(h+1)\bar{K}(\tau,\tau_0)<1$, where λ and θ are set as the same in Theorem 3.1.

It is easy to see that (40) is fulfilled with

$$c_1=\min_{i\in S}\lambda_{\min}(Q_i),\ c_2=\max_{i\in S}\lambda_{\max}(Q_i).$$

If we set $\lambda_4=2\max_{i\in S}\|Q_i\|$, then (19) is fulfilled when choose $0<\lambda_1<\lambda_3/\lambda_4^2$ and $0<\lambda_2=\lambda_3-\lambda_1\lambda_4^2$. Therefore the Corollary 4.2 appears naturally.

5 Example

Example 5.1 Let us consider a 2-dimension linear hybrid system

$$\mathrm{d}x(t)=A(r(t))x(t)\mathrm{d}t+B(r(t))x(t)\mathrm{d}B(t), \tag{55}$$

for $t\geqslant t_0$. Here $B(t)$ is a scalar Brownian motion. $r(t)$ is a two state Markov chain with its space on $S=\{1,2\}$, and the generator is

$$\varGamma=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix},$$

and

$$A_1=\begin{bmatrix}1 & 3\\ 4 & -5\end{bmatrix},\ A_2=\begin{bmatrix}-3 & 4\\ 5 & 2\end{bmatrix},$$
$$B_1=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix},\ B_2=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

It can be seen from Fig.1 that the original system (55) is not almost surely exponentially stable.

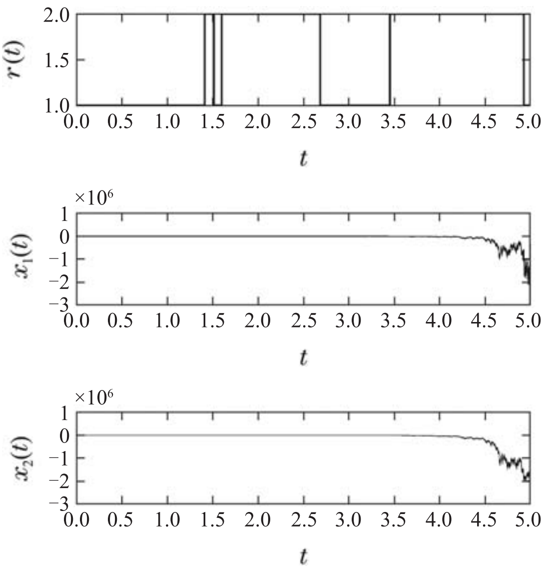

Fig. 1 Original system

Fig.1 shows the paths simulation of $r(t)$, $x_1(t)$ and $x_2(t)$ for the original system (55). Here we use the Euler-Maruyama method, where the initial data is $r(0)=1$, $x_1(0)=-2$ and $x_2(0)=1$, the time step is 10^{-6}.

Assume that we observe the state with a time delay τ_0, which means we can only observe the state $x(t-\tau_0)$ at time t. In this situation, we consider a linear feedback control function $u(x,i,t)=F_iG_ix$ to stabilize the system due to the linear coefficients of (55) based on the discrete-time observations of the state. Thus the controlled system can be regarded as a hybrid SDDE as follow:

$$\mathrm{d}x(t)=[A(r(t))x(t)+F(r(t))G(r(t))x(\delta(t))]\mathrm{d}t+$$
$$B(r(t))x(t)\mathrm{d}B(t). \tag{56}$$

Here, we consider the situation of state feedback, i.e. assume that we know $G_1=(1,1),\ G_2=(1,1)$. Now we need to design F_1 and F_2 in $\mathbb{R}^{2\times1}$ and find the proper condition on τ so that the controlled system to be both mean square and almost surely exponentially stable. It is easy to see that Assumptions 2.1 and 2.2 hold with $K_1=6.672$ and $K_3=2.289$. Define the left-hand-side term of (54) by

$$\bar{Q}_i:=Q_i(A_i+F_iG_i)+(A_i^{\mathrm{T}}+G_i^{\mathrm{T}}F_i^{\mathrm{T}})Q_i+$$
$$B_i^{\mathrm{T}}Q_iB_i+\sum_{j=1}^{2}\gamma_{ij}Q_j.$$

Then using Corollary 4.2 we can verify that $Q_1=Q_2$

$= I$ and

$$F_1 = \begin{bmatrix} -7 \\ -2 \end{bmatrix}, \; F_2 = \begin{bmatrix} -3 \\ -8 \end{bmatrix}, \tag{57}$$

which implies

$$\bar{Q}_1 = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}, \; \bar{Q}_2 = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}. \tag{58}$$

That is $x^{\mathrm{T}}\bar{Q}_i x \leqslant -10|x|^2$. Which makes $\lambda_3 = 10$ and $K_2 = 12.083$. One can verify that (40) is fulfilled with $c_1 = c_2 = 1$ and $\lambda_4 = 2$, $\lambda_1 = 1.25$, and $\lambda_2 = 5$. Therefore, condition (21) and (22) becomes $6.25 > 146(\tau+\tau_0)(673.02(\tau+\tau_0)+5.24)$, $\tau+\tau_0 \leqslant 1/49$ and $2(h+1)\bar{K}(\tau,\tau_0) < 1$ which means $\tau + \tau_0 < 0.0049$. So let us set F_i as (57), then the controlled hybrid system (56) is both mean square and almost surely exponentially stable if $\tau + \tau_0 < 0.0049$ by Corollary 4.2. The simulations in Fig.2 are in line with the theoretical result. It should be mentioned that the existing result in Ref. [36] ask for $\tau + \tau_0 < 0.00003$, but our new result only needs $\tau + \tau_0 < 0.0049$. This indicates that we have improved the existing result significantly.

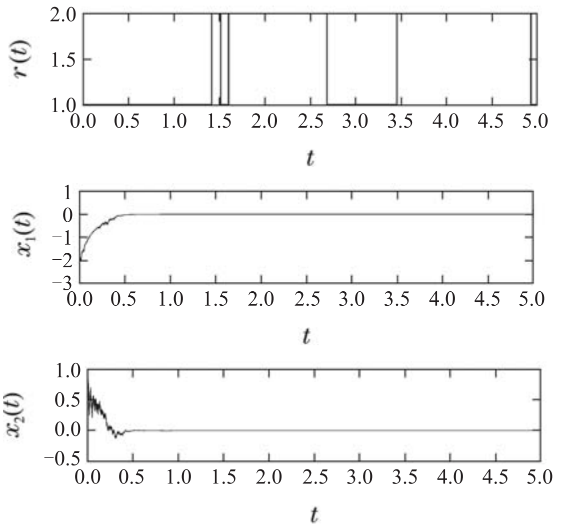

Fig. 2 Control system

Fig.2 shows the paths simulation of $r(t)$, $x_1(t)$ and $x_2(t)$ for the controlled system (56) with $\tau_0 = 0.002$ and $\tau = 0.0029$. Here we use the Euler-Maruyama method, where the initial data is still $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$, and the time step is 10^{-6} as well.

6 Conclusions and future research

This paper studies the stabilisation of hybrid SDEs by the time-delay feedback control based on discrete-time state observation. Making use of Lyapunov functional, we discussed the stabilization in the sense of H_∞-stability, asymptotically stability and exponential stability, and get a significantly improved bound on both τ and τ_0. An interesting question is if not only the stat observations $x(t)$ have time delay but also the Markov chain $r(t)$, which means the control function becomes $u(x(\delta(t)), r(\delta(t)), t)$, what is the stabilization theory in such case. This could be one of the future research.

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Appendix Proof of Lemma 3.2

Proof One may notice that $x(t)$ has different forms when $t < 0$ and $t \geqslant 0$. This is because of that the initial data of the controlled system (6) come from the observations of the original system (5). More precisely, $x(t)$ fulfills (6) when $t \geqslant 0$ and (8) when $t < 0$. Therefore, $x(\delta(t))$ also has different forms when $0 \leqslant t < h\tau$ and $t \geqslant h\tau$, where h is denoted by (9). For the first part, let $n \geqslant h$ be an integer, for any $t \in [n\tau, (n+1)\tau)$ we have $\delta(t) = n\tau - \tau_0 \geqslant 0$. By the same techniques in [26], it is not hard to obtain that for any $t \in [n\tau, (n+1)\tau)$

$$\mathrm{E}|x(t) - x(\delta(t))|^2 \leqslant K(\tau, \tau_0) \sum_{k=0}^{h} \mathrm{E}|x((n-k)\tau - \tau_0)|^2, \tag{A1}$$

where $K(\tau, \tau_0)$ is define by (33). Then for the second part, let $0 \leqslant n \leqslant h-1$ be an integer, for any $t \in [n\tau, (n+1)\tau)$ we have $\delta(t) = n\tau - \tau_0 < 0$. Then we have

$$x(t) - x(\delta(t)) = x(t) - x(0) + x(0) - x(\delta(t)) = \int_{n\tau-\tau_0}^{t} f(x(s), r(s), s)\mathrm{d}s + \int_{n\tau-\tau_0}^{t} g(x(s), r(s), s)\mathrm{d}B(s) + \int_0^t u(x(\delta(s)), r(s), s)\mathrm{d}s.$$

It is not hard to get the same property (A1) for $0 \leqslant t < h\tau$ by using the same technique, together with the first part, we obtain that (A1) holds for any $t \geqslant 0$. Then we can get

$$\mathrm{e}^{\alpha t}\mathrm{E}|x(t) - x(\delta(t))|^2 \leqslant \mathrm{e}^{\alpha t} K(\tau, \tau_0) \sum_{k=0}^{h} \mathrm{E}|x((n-k)\tau - \tau_0)|^2. \tag{A2}$$

By dividing the interval $[0, t]$ into subintervals, it follows

$$\int_0^t \mathrm{e}^{\alpha s}\mathrm{E}|x(s) - x(\delta(s))|^2\mathrm{d}s = \int_{n\tau}^t \mathrm{e}^{\alpha s}\mathrm{E}|x(s) - x(\delta(s))|^2\mathrm{d}s + \sum_{l=0}^{n-1} \int_{(n-l-1)\tau}^{(n-l)\tau} \mathrm{e}^{\alpha s}\mathrm{E}|x(s) - x(\delta(s))|^2\mathrm{d}s,$$

for any $t \geqslant 0$. Applying (A2) to each subinterval, we have

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\alpha s}\mathrm{E}|x(s) - x(\delta(s))|^2\mathrm{d}s \leqslant \\
&K(\tau, \tau_0) \int_{n\tau}^t \mathrm{e}^{\alpha s} \sum_{k=0}^{h} \mathrm{E}|x((n-k)\tau - \tau_0)|^2\mathrm{d}s + \\
&K(\tau, \tau_0) \sum_{l=0}^{n-1} \int_{(n-l-1)}^{(n-l)\tau} \tau\mathrm{e}^{\alpha s} \sum_{k=n-h-1-l}^{n-1-l} \mathrm{E}|x(k\tau - \tau_0)|^2\mathrm{d}s \leqslant \\
&K(\tau, \tau_0) \sum_{k=0}^{h} \mathrm{e}^{\alpha k\tau} \int_{(n-k)\tau}^{t-k\tau} \mathrm{e}^{\alpha s}\mathrm{E}|x(\delta(s))|^2\mathrm{d}s + \\
&K(\tau, \tau_0) \sum_{l=0}^{n-1} \sum_{k=0}^{h} \mathrm{e}^{\alpha k\tau} \int_{(n-1-k-l)\tau}^{(n-k-l)} \tau\mathrm{e}^{\alpha s}\mathrm{E}|x(\delta(s))|^2\mathrm{d}s.
\end{aligned}$$

Then the desired assertion (35) holds by combining the corresponding integrals

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\alpha s}\mathrm{E}|x(s) - x(\delta(s))|^2\mathrm{d}s \leqslant \\
&K(\tau, \tau_0) \sum_{k=0}^{h} \mathrm{e}^{\alpha k\tau} \int_{-k\tau}^{t-k\tau} \mathrm{e}^{\alpha s}\mathrm{E}|x(\delta(s))|^2\mathrm{d}s \leqslant \\
&(h+1)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau} \int_0^t \mathrm{e}^{\alpha s}\mathrm{E}|x(\delta(s))|^2\mathrm{d}s + \\
&K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau} \sum_{k=1}^{h} \int_{-k\tau}^{0} \mathrm{e}^{\alpha s}\mathrm{E}|x(\delta(s))|^2\mathrm{d}s \leqslant \\
&2(h+1)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau} \\
&\int_0^t \mathrm{e}^{\alpha s}(\mathrm{E}|x(s) - x(\delta(s))|^2 + \mathrm{E}|x(s)|^2)\mathrm{d}s + \\
&\frac{\tau}{2}(h+h^2)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau} \sup_{\theta\in[-h\tau-\tau_0, 0]} \mathrm{E}|x(\theta)|^2 \leqslant \\
&\frac{2(h+1)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau}}{1 - 2(h+1)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau}} \int_0^t \mathrm{e}^{\alpha s}\mathrm{E}|x(s)|^2\mathrm{d}s + \\
&\frac{\tau(h+h^2)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau}}{2 - 4(h+1)K(\tau, \tau_0)\mathrm{e}^{\alpha h\tau}} \sup_{\theta\in[-h\tau-\tau_0, 0]} \mathrm{E}|x(\theta)|^2.
\end{aligned}$$

The proof is complete due to

$$\sup_{\theta\in[-h\tau-\tau_0, 0]} \mathrm{E}|x(\theta)|^2 = \sup_{\theta\in[-\tau_0, 0]} \mathrm{E}|x(\theta)|^2.$$

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