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具Cauchy-Ventcel边界的阻尼波方程的对数衰减性

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摘要: 本文研究有界区域上具Cauchy-Ventcel边界条件的波动方程的解的衰减性质. 在不要求耗散区域满足几何控制条件的情形下, 得到了波方程的对数衰减结果. 主要结果的证明依赖于具Cauchy-Ventcel边界条件的椭圆方程的插值不等式以及关于该椭圆方程的预解式估计. 为得到期望的插值不等式, 本文采用的工具是整体Carleman估计.

关键词: 对数衰减; 波方程; Cauchy-Ventel边界; Carleman估计

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Logarithmic decay of wave equations with Cauchy-Ventcel boundary conditions

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Abstract: This paper is devoted to a study of decay properties for a class of wave equations with Cauchy-Ventcel boundary conditions and a local internal damping. Based on an estimate on the resolvent operator, solutions of the wave equations under consideration are proved to decay logarithmically without any geometric control condition. The proof of the decay result relies on the interpolation inequalities for an elliptic equation with Cauchy-Ventcel boundary conditions and the estimate of the resolvent operator for that equation. The main tool to derive the desired interpolation inequality is global Carleman estimate.

Key words: Logarithmic decay; wave equations; Cauchy-Ventcel boundary condition; Carleman estimate

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1 Introduction

Let $\varOmega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with boundary $\varGamma$ of class C^2, and $\varGamma_1$ and $\varGamma_2$ be two open subsets of $\varGamma$, such that $\varGamma = \varGamma_1 \cup \varGamma_2$ and $\bar{\varGamma}_1 \cap \bar{\varGamma}_2 = \varnothing$. Set $\mathbb{R}^+ = [0, \infty)$. Denote by $\bar{c}$ the complex conjugate of a complex number $c \in \mathbb{C}$ and by i the imaginary unit.

Consider the following wave equation with an internal damping:

$$\begin{cases} u_{tt} - \Delta u + d(x)u_t = 0, & \mathbb{R}^+ \times \varOmega, \\ \dfrac{\partial u}{\partial \nu} + p(x)u - \varDelta_{\mathrm{T}} u = 0, & \mathbb{R}^+ \times \varGamma_1, \\ u = 0, & \mathbb{R}^+ \times \varGamma_2, \\ u(0) = u^0,\ u_t(0) = u^1, & \varOmega, \end{cases} \tag{1}$$

where $p \in C^1(\varGamma_1; \mathbb{R}^+)$, $d \in L^\infty(\varOmega; \mathbb{R}^+)$, $\varDelta_{\mathrm{T}}$ denotes the Laplace-Beltrami operator on $\varGamma_1$ and $\nu = (\nu_1, \cdots, \nu_n)$ denotes the outward unit normal vector of $\varOmega$.

The wave equation (1) possesses a Ventcel boundary condition on $\varGamma_1$. Such boundary condition was first introduced by Ventcel in [1] for second-order elliptic equations. The associated equation may model the heat exchange between a solid $\varOmega$ and its environment, when the boundary $\varGamma_1$ is covered with a thin layer of material with high conductibility (see [2]), or model the behavior of an elastic body covered by a thin shell of high rigidity (see [3–4]).

As far as we know, there are numerous works on de-

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cay properties of wave equations. For example, Bardos-Lebeau-Rauch (see [5]) showed that under suitable conditions, the energy of solutions for wave equations decays exponentially, if and only if the effective damping domain satisfies the geometric control conditions. We refer to [6–7] and rich references therein for some known exponential decay results for the wave equations with Dirichlet or Neumann boundary conditions. In [9–10], the polynomial decay for wave equations with Dirichlet boundary conditions under some special geometric conditions was studied. When the geometric control condition fails, the logarithmic decay of wave equations was first considered by Lebeau in [11]. Later, [12] was devoted to a system with a Neumann dissipative term on the boundary. In [13], Bellassoued derived the logarithmic decay result for an elastic wave equation with Neumann dissipative term on the boundary and an internal damping, respectively. Cornilleau-Robbiano[14] considered the logarithmic decay for wave equations with Zaremba boundary conditions.

There are also many known works on the longtime behavior of evolution equations with Ventcel boundary conditions. For example, in [3], the boundary stabilization of a linear elastodynamic system with dynamic Ventcel boundary conditions was studied. Not only an exponential stabilization result was obtained by choosing a suitable boundary feedback, but also a negative result was shown for the wave equation with stationary Ventcel boundary conditions, even if a linear frictional damping was applied to the whole boundary. In [15], the uniform energy decay result for wave equations with internal damping and stationary Ventcel boundary conditions were derived. Later, this result was improved in [16] by relaxing the geometric conditions on damping. Recently, in [17], the stabilization for damped wave equations with Ventcel boundary conditions on a smooth Riemannian manifold was studied. Furthermore, we refer to [18–19] for some known polynomial decay results. However, as far as we know, there is not any known logarithmic decay result for damped wave equation with Cauchy-Ventcel boundary conditions. In this paper, the decay property of the equation (1) is proved by establishing an estimate for the resolvent operator.

As one of important methods, the decay property of solutions to evolution equations may be reduced to a suitable resolvent estimate for the associated semigroup generator (see [8]). In this respect, we refer to [20–21] for some known results on the exponential decay, [22–23] on the polynomial decay and [11–12,14] on the logarithmic decay, respectively. In [24] and references therein, there are some characterization of decay rates on solutions of abstract evolution equations. Based on this method, the key of proving the logarithmic decay result of this paper is to establish the associated resolvent estimate. For this purpose, a global Carleman estimate for elliptic equations with Ventcel boundary conditions is derived.

The rest of this paper is organized as follows. Section 2 is devoted to stating the main result of this paper. In Section 3, a global Carleman estimates for elliptic equations with Ventcel boundary conditions is proved, based on which an interpolation inequality for elliptic equations is established in Section 4. Section 5 is devoted to the proof of the main result.

2 Main results

In this section, we present the logarithmic decay result for the damped wave equation (1) with Cauchy-Ventcel boundary conditions. To this aim, introduce the following assumptions on the damping coefficient:

A) There exist a nonempty open subset ω^* of Ω and $c_0 > 0$, such that $d(x) \geqslant c_0$, for a. e. $x \in \omega^*$.

Also, set the Hilbert space

$$V = \{u \in H^1(\Omega) | u|_{\Gamma_2} = 0,\ u|_{\Gamma_1} \in H^1(\Gamma_1)\},$$

endowed with the topology

$$\|u\|_{\mathrm{V}}^2 = \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{\Gamma_1} |\nabla_{\mathrm{T}} u|^2 \mathrm{d}\Gamma, \tag{2}$$

where ∇_{T} denotes the tangential derivative along Γ_1. Put $H = V \times L^2(\Omega)$ and define an unbounded operator $A : \mathcal{D}(A) \subset H \to H$ by

$$\mathcal{D}(A) = \{U \in H | AU \in H,\ (\frac{\partial u}{\partial \nu} + pu - \Delta_{\mathrm{T}} u)|_{\Gamma_1} = 0\}$$

with $U = (u, v)$ and $AU = (v, \Delta u - d(x)v)$. It is easy to check that A generates a C_0-semigroup $\{\mathrm{e}^{tA}\}_{t\in\mathbb{R}^+}$ on H. Without causing confusion, we use the same notation for real valued and complex valued function spaces. Moreover, for the equation (1), define its energy as follows:

$$E(t) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |u_t|^2)\mathrm{d}x + \frac{1}{2}\int_{\Gamma_1} (p|u|^2 + |\nabla_{\mathrm{T}} u|^2)\mathrm{d}\Gamma.$$

Recall the surface divergence theorem (see e.g. [25]):

$$\int_{\Gamma_1} \Delta_{\mathrm{T}} u v \mathrm{d}\Gamma = -\int_{\Gamma_1} \nabla_{\mathrm{T}} u \cdot \nabla_{\mathrm{T}} v \mathrm{d}\Gamma,\quad \forall u \in H^2(\Gamma_1),\ v \in H^1(\Gamma_1). \tag{3}$$

By (2)–(3), for any $t_2 > t_1 \geqslant 0$, it follows that

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_\Omega d(x)|u_t|^2 \mathrm{d}x\mathrm{d}t \leqslant 0.$$

Therefore, by LaSalle's invariance principle ([26, p. 18]), the energy of solutions to (1) decays to zero, without any geometric control condition.

The main result of this paper is stated as follows:

Theorem 1 Assume that (A) holds. Then there exists a positive constant C, such that for any $(u^0, u^1) \in \mathcal{D}(A)$, the associated solution $(u, u_t) = \mathrm{e}^{tA}(u^0, u^1) \in$

$C(\mathbb{R}^+;\mathcal{D}(A))\cap C^1(\mathbb{R}^+;H)$ of the equation (1) satisfies

$$\|\mathrm{e}^{tA}(u^0,u^1)\|_H \leqslant \frac{C}{\ln(2+t)}\|(u^0,u^1)\|_{\mathcal{D}(A)}. \quad (4)$$

Throughout this paper, C denotes a generic positive constant, which may be different from line to line. Recall that the resolvent estimates for abstract evolution equations indeed imply the decay properties for solutions of the equations.

Lemma 1[24] Let H be a Hilbert space. Assume that A generates a bounded C_0-semigroup on H. If $M(\xi)\leqslant C\mathrm{e}^{C\xi}$ for a positive constant C and any $\xi\geqslant 0$, then for any $k\geqslant 1$, there is a $C_k>0$, such that

$$m_k(t)\leqslant \frac{C_k}{\ln^k(2+t)},$$

where

$$m_k(t)=\sup_{s\geqslant t}\|\mathrm{e}^{sA}(I-A)^{-k}\|$$

and

$$M(\xi)=\sup_{|\tau|\leqslant\xi}\|(i\tau I-A)^{-1}\|.$$

Denote by $\rho(A)$ the resolvent set of A. Then by Lemma 1, in order to derive the desired decay rate in Theorem 1, it suffices to prove the following estimate for the resolvent operator.

Theorem 2 Under the assumption of Theorem 1, $i\mathbb{R}\subset\rho(A)$, and there exists a positive constant C, such that

$$\|(A-i\beta I)^{-1}\|_{\mathcal{L}(H)}\leqslant C\mathrm{e}^{C|\beta|},\ \forall\,\beta\in\mathbb{R}. \quad (5)$$

3 Global Carleman estimate for elliptic equations with Ventcel boundary conditions

Set $Q=(-2,2)\times\Omega$, $\Sigma=(-2,2)\times\Gamma$ and $\Sigma_j=(-2,2)\times\Gamma_j$, for $j=1,2$. Consider the following elliptic equation:

$$\begin{cases} z_{\mathrm{ss}}+\Delta z+id(x)z_{\mathrm{s}}=z_0, & Q,\\ \dfrac{\partial z}{\partial\nu}+p(x)z-\Delta_{\mathrm{T}}z=0, & \Sigma_1,\\ z=0, & \Sigma_2,\end{cases} \quad (6)$$

where $z_0\in L^2(Q)$.

Before giving weight functions, we first recall the following known result.

Lemma 2[27] Let ω_0 be any given subdomain of Ω satisfying $\bar{\omega}_0\subset\omega^*$. Then there exists a function $\hat{\psi}\in C^2(\bar{\Omega})$, such that

$$\hat{\psi}>0 \text{ in } \Omega,\ \hat{\psi}=0 \text{ on } \Gamma \text{ and } |\nabla\hat{\psi}|>0 \text{ in } \overline{\Omega\setminus\omega_0}. \quad (7)$$

Remark 1 Since $\nabla\hat{\psi}=\nabla_{\mathrm{T}}\hat{\psi}+\dfrac{\partial\hat{\psi}}{\partial\nu}\nu$ on Γ, $\hat{\psi}$ in Lemma 2 satisfies

$$\nabla_{\mathrm{T}}\hat{\psi}=0,\ |\nabla\hat{\psi}|=|\frac{\partial\hat{\psi}}{\partial\nu}| \text{ and } \frac{\partial\hat{\psi}}{\partial\nu}\leqslant c<0 \text{ on } \Gamma,$$

for a negative constant c. Furthermore, define two constants

$$b=\sqrt{1+\frac{\ln(2+\mathrm{e}^{\mu})}{\mu}},\ b_0=\sqrt{b^2-\frac{1}{\mu}\ln(\frac{1+\mathrm{e}^{\mu}}{\mathrm{e}^{\mu}})},$$

where $\mu>\ln 2$ is sufficiently large, such that $1<b_0<b\leqslant 2$. For parameters $\lambda,\mu\geqslant 1$, define the weight functions θ and ϕ as follows:

$$\begin{cases}\theta=\mathrm{e}^{\ell},\ \ell=\lambda\phi,\ \phi=\mathrm{e}^{\mu\psi},\\ \psi(s,x)=\dfrac{\hat{\psi}(x)}{\|\hat{\psi}\|_{L^\infty(\Omega)}}+b^2-s^2.\end{cases} \quad (8)$$

Then we have the following Carleman estimate for (6).

Theorem 3 Assume that $z\in H^2(Q)$ is a solution to (6) with $z(\pm b,\cdot)=z_{\mathrm{s}}(\pm b,\cdot)=0$ in Ω. If the condition (A) holds, there is a constant $\mu_0>0$, such that for any $\mu\geqslant\mu_0$, one can find two constants $C=C(\mu)>0$ and $\lambda_0=\lambda_0(\mu)$ so that for any $\lambda\geqslant\lambda_0$, the following estimate holds:

$$\lambda\mu^2\int_{-b}^{b}\int_\Omega\theta^2\phi(|\nabla z|^2+|z_{\mathrm{s}}|^2+\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s+$$
$$\lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s\leqslant$$
$$C[\|\theta z_0\|^2_{L^2(Q)}+\mathrm{e}^{C\lambda}\int_{-b}^{b}\int_{\omega^*}|z|^2\mathrm{d}x\mathrm{d}s]. \quad (9)$$

Remark 2 The proof of Theorem 3 is based on a point-wise weighted estimate, which will be given later. On the other hand, note however that we need to consider the problem with nonhomogeneous Cauchy-Ventcel boundary. Hence, the treatment on the corresponding boundary terms become much more complicated than the usual case with homogeneous Dirichlet boundary condition.

Now, we recall the following weighted inequality for elliptic operators, which can be obtained immediately from [28, Theorem 3.1].

Lemma 3 Assume that $z\in H^1((-b,b)\times\Omega)$ satisfies that $z_{\mathrm{ss}}+\Delta z=f$ (in the sense of distribution) with $f\in L^2((-b,b)\times\Omega)$ and $z(\pm b,\cdot)=z_{\mathrm{s}}(\pm b,\cdot)=0$ in Ω. Then there is a constant $\mu_1>0$, such that for any $\mu\geqslant\mu_1$, one can find two constants $C=C(\mu)>0$ and $\lambda_1=\lambda_1(\mu)$ so that for any $\lambda\geqslant\lambda_1$, it holds that

$$\lambda\mu^2\int_{-b}^{b}\int_\Omega\theta^2\phi|\nabla\psi|^2(|z_{\mathrm{s}}|^2+|\nabla z|^2+$$
$$\lambda^2\mu^2\phi^2|\nabla\psi|^2|z|^2)\mathrm{d}x\mathrm{d}s\leqslant$$
$$C[\int_{-b}^{b}\int_\Omega\theta^2|f|^2\mathrm{d}x\mathrm{d}s+$$
$$\int_{-b}^{b}\int_\Gamma\sum_{k=1}^{n}V^k\cdot\nu_k\mathrm{d}\Gamma\mathrm{d}s+$$
$$\lambda\mu\int_{-b}^{b}\int_\Omega\theta^2\phi(|z_{\mathrm{s}}|^2+|\nabla z|^2+\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s], \quad (10)$$

where

$$\begin{cases} A = \ell_{\mathrm{s}}^2 + \ell_{\mathrm{ss}} + |\nabla \ell|^2 + \Delta \ell,\ v = \theta z, \\ V^k = [2\ell_{\mathrm{s}}(v_{x_k}\bar{v}_s + \bar{v}_{x_k}v_s) + \\ \qquad 2(\ell_{\mathrm{ss}} + \Delta\ell)(v_{x_k}\bar{v} + \bar{v}_{x_k}v) - \\ \qquad 2\ell_{x_k}|v_{\mathrm{s}}|^2 + 2A\ell_{x_k}|v|^2 + \\ \qquad 2\sum\limits_{j=1}^{n}\ell_{x_j}(v_{x_j}\bar{v}_{x_k} + \bar{v}_{x_j}v_{x_k}) - 2\ell_{x_k}|\nabla v|^2]. \end{cases} \tag{11}$$

Now, we give a proof of Theorem 3.

Proof The whole proof is divided into four parts.

Step 1 In order to deal with estimates on boundary terms, we choose another two weight functions $\tilde{\theta}$ and $\tilde{\phi}$:

$$\begin{cases} \tilde{\theta} = \mathrm{e}^{\tilde{\ell}},\ \tilde{\ell} = \lambda\tilde{\phi},\ \tilde{\phi} = \mathrm{e}^{\mu\tilde{\psi}} \\ \tilde{\psi} = -\dfrac{\hat{\psi}(x)}{\|\hat{\psi}\|_{L^\infty(\Omega)}} + b^2 - s^2. \end{cases} \tag{12}$$

It is easy to check that

$$\tilde{\psi} \leqslant \psi,\ 0 < \tilde{\phi} \leqslant \phi,\ 0 < \tilde{\theta} \leqslant \theta. \tag{13}$$

Now, we apply Lemma 3 to (6). By (13) and $z(\pm b, x) = z_{\mathrm{s}}(\pm b, x) = 0$, there is a constant $\mu_1 > 0$, such that for all $\mu \geqslant \mu_1$, one can find $\lambda_1 = \lambda_1(\mu)$ so that for any $\lambda \geqslant \lambda_1$,

$$\begin{aligned} &\lambda\mu^2\int_{-b}^{b}\int_{\Omega}\theta^2\phi|\nabla\psi|^2(|z_{\mathrm{s}}|^2 + |\nabla z|^2 + \\ &\qquad \lambda^2\mu^2\phi^2|\nabla\psi|^2|z|^2)\mathrm{d}x\mathrm{d}s \leqslant \\ &C[\|\theta z_0\|^2_{L^2(Q)} + \int_{-b}^{b}\int_{\Gamma}\sum_{k=1}^{n}(V^k + \tilde{V}^k)\cdot\nu_k\mathrm{d}\Gamma\mathrm{d}s + \\ &\lambda\mu\int_{-b}^{b}\int_{\Omega}\theta^2\phi(|z_{\mathrm{s}}|^2 + |\nabla z|^2 + \\ &\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s], \end{aligned} \tag{14}$$

where $\tilde{V}^k$ has the same form with V^k, only in which, ℓ is replaced by $\tilde{\ell}$.

Step 2 We estimate "$\int_{-b}^{b}\int_{\Gamma}\sum_{k=1}^{n}(V^k + \tilde{V}^k)\cdot\nu_k\mathrm{d}\Gamma\mathrm{d}s$".

By (7)–(8) and (12), the following equalities hold on Γ:

$$\begin{cases} \phi = \tilde{\phi},\ \ell = \tilde{\ell},\ \theta = \tilde{\theta}, \\ \ell_{\mathrm{s}} = \lambda\mu\phi\psi_{\mathrm{s}} = \lambda\mu\tilde{\phi}\tilde{\psi}_{\mathrm{s}} = \tilde{\ell}_{\mathrm{s}}, \\ \sum\limits_{j=1}^{n}(\ell_{x_j} + \tilde{\ell}_{x_j})\nu_j = \lambda\mu\phi\sum\limits_{j=1}^{n}(\psi_{x_j} + \tilde{\psi}_{x_j})\nu_j = 0. \end{cases} \tag{15}$$

Noting that $z = 0$ on Γ_2. Then by (15), it is easy to first show that

$$\int_{-b}^{b}\int_{\Gamma_2}\sum_{k=1}^{n}(V^k + \tilde{V}^k)\cdot\nu_{x_k}\mathrm{d}x\mathrm{d}s = 0. \tag{16}$$

In the following, we give an estimate on

$$\int_{-b}^{b}\int_{\Gamma_1}\sum_{k=1}^{n}(V^k + \tilde{V}^k)\cdot\nu_k\mathrm{d}\Gamma\mathrm{d}s.$$

To this aim, denote by H_j^k and $\tilde{H}_j^k$ $(j = 1, 2, \cdots, 6)$ six terms in the right side of V^k and $\tilde{V}^k$, respectively. Noting that $v = \theta z$ and $\tilde{v} = \tilde{\theta}z$ on the boundary Γ_1, we have that

$$\begin{aligned} &(H_1^k + \tilde{H}_1^k)\cdot\nu_k = \\ &2[\ell_{\mathrm{s}}(v_{x_k}\bar{v}_s + \bar{v}_{x_k}v_s) + \tilde{\ell}_{\mathrm{s}}(\tilde{v}_{x_k}\bar{\tilde{v}}_s + \bar{\tilde{v}}_{x_k}\tilde{v}_s)]\cdot\nu_k = \\ &4\theta^2\ell_{\mathrm{s}}(\frac{\partial z}{\partial\nu}\bar{z}_{\mathrm{s}} + \frac{\partial\bar{z}}{\partial\nu}z_{\mathrm{s}}) + 4\theta^2\ell_{\mathrm{s}}^2(\frac{\partial z}{\partial\nu}\bar{z} + \frac{\partial\bar{z}}{\partial\nu}z) = \\ &4\theta^2\ell_{\mathrm{s}}(\Delta_{\mathrm{T}}z\bar{z}_{\mathrm{s}} + \Delta_{\mathrm{T}}\bar{z}z_{\mathrm{s}}) - 4(p(x)\theta^2\ell_{\mathrm{s}}|z|^2)_{\mathrm{s}} + \\ &4\theta^2\ell_{\mathrm{s}}^2(\Delta_{\mathrm{T}}z\bar{z} + \Delta_{\mathrm{T}}\bar{z}z) + 4p(x)\theta^2\ell_{\mathrm{ss}}|z|^2. \end{aligned}$$

Further,

$$\begin{aligned} &(H_2^k + \tilde{H}_2^k)\cdot\nu_k + (H_3^k + \tilde{H}_3^k)\cdot\nu_k + \\ &(H_4^k + \tilde{H}_4^k)\cdot\nu_k = \\ &2[(\ell_{\mathrm{ss}} + \Delta\ell)(v_{x_k}\bar{v} + \bar{v}_{x_k}v) + \\ &(\tilde{\ell}_{\mathrm{ss}} + \Delta\tilde{\ell})(\tilde{v}_{x_k}\bar{\tilde{v}} + \bar{\tilde{v}}_{x_k}\tilde{v})]\cdot\nu_k - \\ &2[\ell_{x_k}|v_{\mathrm{s}}|^2 + \tilde{\ell}_{x_k}|\tilde{v}_{\mathrm{s}}|^2 - (A - \tilde{A})\theta^2\ell_{x_k}|z|^2]\cdot\nu_k = \\ &2(2\ell_{\mathrm{ss}} + \Delta\ell + \Delta\tilde{\ell})\theta^2(\frac{\partial z}{\partial\nu}\bar{z} + \frac{\partial\bar{z}}{\partial\nu}z) + \\ &6\theta^2\lambda\mu\phi\frac{\partial\psi}{\partial\nu}(\Delta\ell - \Delta\tilde{\ell})|z|^2. \end{aligned}$$

Noting that $\psi = 0$ on the boundary, we have that

$$\begin{aligned} &(H_5^k + \tilde{H}_5^k)\cdot\nu_k + (H_6^k + \tilde{H}_6^k)\cdot\nu_k = \\ &[2\sum_{j=1}^{n}\ell_{x_j}(v_{x_j}\bar{v}_{x_k} + \bar{v}_{x_j}v_{x_k}) - 2\ell_{x_k}|\nabla v|^2]\cdot\nu_k + \\ &2\sum_{j=1}^{n}\tilde{\ell}_{x_j}(\tilde{v}_{x_j}\bar{\tilde{v}}_{x_k} + \bar{\tilde{v}}_{x_j}\tilde{v}_{x_k}) - 2\tilde{\ell}_{x_k}|\nabla\tilde{v}|^2]\cdot\nu_k = \\ &4\theta^2|\nabla\ell|^2(\frac{\partial z}{\partial\nu}\bar{z} + \frac{\partial\bar{z}}{\partial\nu}z) = \\ &4\theta^2|\nabla\ell|^2(\Delta_{\mathrm{T}}z\bar{z} + \Delta_{\mathrm{T}}\bar{z}z) - 8p(x)\theta^2|\nabla\ell|^2|z|^2. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &\int_{-b}^{b}\int_{\Gamma_1}\sum_{k=1}^{n}(V^k + \tilde{V}^k)\cdot\nu_k\mathrm{d}\Gamma\mathrm{d}s \leqslant \\ &\int_{-b}^{b}\int_{\Gamma_1}[4\theta^2\ell_{\mathrm{s}}(\Delta_{\mathrm{T}}z\bar{z}_{\mathrm{s}} + \Delta_{\mathrm{T}}\bar{z}z_{\mathrm{s}}) + \\ &2E\theta^2(\Delta_{\mathrm{T}}z\bar{z} + \Delta_{\mathrm{T}}\bar{z}z) + F\theta^2|z|^2]\mathrm{d}\Gamma\mathrm{d}s, \end{aligned} \tag{17}$$

where

$$\begin{cases} E = 2\ell_{\mathrm{s}}^2 + 2|\nabla\ell|^2 + (2\ell_{\mathrm{ss}} + \Delta\ell + \Delta\tilde{\ell}), \\ F = 4p(x)\ell_{\mathrm{ss}} - 4p(x)(2\ell_{\mathrm{ss}} + \Delta\ell + \Delta\tilde{\ell}) + \\ \qquad 6\lambda\mu\phi\dfrac{\partial\psi}{\partial\nu}(\Delta\ell - \Delta\tilde{\ell}) - 8p(x)|\nabla\ell|^2. \end{cases} \tag{18}$$

Noting that $\nabla_{\mathrm{T}}\hat{\psi} = 0$ on Γ_1 in Remark 1 and using the surface divergence theorem, we have

$$\begin{aligned} &4\int_{-b}^{b}\int_{\Gamma_1}\theta^2\ell_{\mathrm{s}}(\Delta_{\mathrm{T}}z\bar{z}_{\mathrm{s}} + \Delta_{\mathrm{T}}\bar{z}z_{\mathrm{s}})\mathrm{d}\Gamma\mathrm{d}s = \\ &-4\int_{-b}^{b}\int_{\Gamma_1}\theta^2\ell_{\mathrm{s}}(\nabla_{\mathrm{T}}z\cdot\nabla_{\mathrm{T}}\bar{z}_{\mathrm{s}} + \nabla_{\mathrm{T}}\bar{z}\cdot\nabla_{\mathrm{T}}z_{\mathrm{s}})\mathrm{d}\Gamma\mathrm{d}s = \\ &4\int_{-b}^{b}\int_{\Gamma_1}(\theta^2\ell_{\mathrm{s}})_{\mathrm{s}}|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s. \end{aligned} \tag{19}$$

Similarly,

$$2\int_{-b}^{b}\int_{\Gamma_1}E\theta^2(\Delta_{\mathrm{T}}z\bar{z} + \Delta_{\mathrm{T}}\bar{z}z)\mathrm{d}\Gamma\mathrm{d}s =$$

$$
\begin{aligned}
&-4\int_{-b}^{b}\int_{\Gamma_1} E\theta^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s-\\
&2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\nabla_{\mathrm{T}}E\cdot(\nabla_{\mathrm{T}}z\bar{z}+\nabla_{\mathrm{T}}\bar{z}z)\mathrm{d}\Gamma\mathrm{d}s\leqslant\\
&-4\int_{-b}^{b}\int_{\Gamma_1} E\theta^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s+\\
&\int_{-b}^{b}\int_{\Gamma_1}\lambda^2\mu^2\phi^2\theta^2(\varepsilon|\nabla_{\mathrm{T}}z|^2+C_\varepsilon|z|^2)\mathrm{d}\Gamma\mathrm{d}s.
\end{aligned}\tag{20}
$$

Noting that $\dfrac{\partial\hat{\psi}}{\partial\nu}\leqslant c<0$, we obtain

$$
\begin{aligned}
&4\int_{-b}^{b}\int_{\Gamma_1}[(\theta^2\ell_{\mathrm{s}})_{\mathrm{s}}-E\theta^2]|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s\leqslant\\
&-c_1\int_{-b}^{b}\int_{\Gamma_1}\theta^2\lambda^2\mu^2\phi^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s
\end{aligned}\tag{21}
$$

for some $c_1>0$.

Now, combining (17)–(21) with (14) and taking ε small enough in (20), we have that

$$
\begin{aligned}
&\lambda\mu^2\int_{-b}^{b}\int_{\Omega}\theta^2\phi|\nabla\psi|^2(|z_{\mathrm{s}}|^2+|\nabla z|^2+\\
&\qquad\lambda^2\mu^2\phi^2|\nabla\psi|^2|z|^2)\mathrm{d}x\mathrm{d}s+\\
&\qquad\lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s\leqslant\\
&C[\|\theta z_0\|^2_{L^2(Q)}+\int_{-b}^{b}\int_{\Gamma_1}\theta^2\lambda^2\mu^2\phi^2|z|^2\mathrm{d}\Gamma\mathrm{d}s+\\
&\lambda\mu\int_{-b}^{b}\int_{\Omega}\theta^2\phi(|z_{\mathrm{s}}|^2+|\nabla z|^2+\\
&\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s].
\end{aligned}\tag{22}
$$

Step 3 Let us estimate "$\lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|z|^2\mathrm{d}\Gamma\mathrm{d}s$". For this purpose, we integrate $\lambda\mu\phi\theta^2\bar{z}\cdot(6)+\lambda\mu\phi\theta^2 z\cdot\overline{(6)}$ on $(-b,b)\times\Omega$. Noting that $z(-b,x)=z(b,x)=0$, by (8) and integration by parts, we get that

$$
\begin{aligned}
&\int_{-b}^{b}\int_{\Omega}\lambda\mu\phi\theta^2(\bar{z}z_0+z\bar{z}_0)\mathrm{d}x\mathrm{d}s+\\
&\int_{-b}^{b}\int_{\Omega}\lambda\mu\phi\theta^2 d(x)[i(\bar{z}z_{\mathrm{s}}-z\bar{z}_{\mathrm{s}})]\mathrm{d}x\mathrm{d}s=\\
&\lambda\mu\int_{-b}^{b}\int_{\Omega}[-2\theta^2\phi|z_{\mathrm{s}}|^2+(\theta^2\phi)_{\mathrm{ss}}|z|^2]\mathrm{d}x\mathrm{d}s+\\
&\lambda\mu\int_{-b}^{b}\int_{\Omega}[-2\theta^2\phi|\nabla z|^2+\Delta(\theta^2\phi)|z|^2]\mathrm{d}x\mathrm{d}s+\\
&\lambda\mu\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi(\bar{z}\Delta_{\mathrm{T}}z+z\Delta_{\mathrm{T}}\bar{z})\mathrm{d}\Gamma\mathrm{d}s+\\
&\lambda\mu\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi[-\frac{\partial\psi}{\partial\nu}(2\lambda\mu\phi+\mu)-\\
&2p(x)]|z|^2\mathrm{d}\Gamma\mathrm{d}s.
\end{aligned}\tag{23}
$$

Further,

$$
\begin{aligned}
&\lambda\mu\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi(\bar{z}\Delta_{\mathrm{T}}z+z\Delta_{\mathrm{T}}\bar{z})\mathrm{d}\Gamma\mathrm{d}s=\\
&-2\lambda\mu\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s.
\end{aligned}
$$

Noting that $\dfrac{\partial\psi}{\partial\nu}\leqslant c<0$ on Γ_1. Hence,

$$
\begin{aligned}
&\lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|z|^2\mathrm{d}\Gamma\mathrm{d}s\leqslant\\
&C[\|\theta z_0\|^2_{L^2(Q)}+\lambda\mu\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s+\\
&\lambda\mu\int_{-b}^{b}\int_{\Omega}\theta^2\phi(|z_{\mathrm{s}}|^2+|\nabla z|^2+\\
&\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s].
\end{aligned}
$$

By (22) and the above estimate,

$$
\begin{aligned}
&\lambda\mu^2\int_{-b}^{b}\int_{\Omega}\theta^2\phi|\nabla\psi|^2(|z_{\mathrm{s}}|^2+\lambda^2\mu^2\phi^2|\nabla\psi|^2|z|^2+\\
&|\nabla z|^2)\mathrm{d}x\mathrm{d}s+\lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s\leqslant\\
&C[\|\theta z_0\|^2_{L^2(Q)}+\lambda\mu\int_{-b}^{b}\int_{\Omega}\theta^2\phi(|z_{\mathrm{s}}|^2+\\
&|\nabla z|^2+\lambda^2\mu^2\phi^2|z|^2)].
\end{aligned}\tag{24}
$$

Moreover, recalling (7) in Lemma 2, we find that

"The first term in the left hand side of (24)" $\geqslant$

$$
\begin{aligned}
&c_2\lambda\mu^2\int_{-b}^{b}\int_{\Omega\backslash\omega_0}\theta^2\phi(|z_{\mathrm{s}}|^2+\\
&\qquad|\nabla z|^2+\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s-\\
&C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|z_{\mathrm{s}}|^2+|\nabla z|^2+\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s,
\end{aligned}
$$

for some positive constant c_2. Hence, it follows that

$$
\begin{aligned}
&\lambda\mu^2\int_{-b}^{b}\int_{\Omega}\theta^2\phi(|\nabla z|^2+|z_{\mathrm{s}}|^2+\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s+\\
&\lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|\nabla_{\mathrm{T}}z|^2\mathrm{d}\Gamma\mathrm{d}s\leqslant\\
&C[\|\theta z_0\|^2_{L^2(Q)}+\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla z|^2+\\
&|z_{\mathrm{s}}|^2+\lambda^2\mu^2\phi^2|z|^2)\mathrm{d}x\mathrm{d}s].
\end{aligned}\tag{25}
$$

Step 4 Let us estimate "$\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi|\nabla z|^2\mathrm{d}x\mathrm{d}s$".

Now, choose a cut-off function $\zeta\in C_0^\infty(\omega^*;[0,1])$ satisfying that $\zeta(x)=1$ on ω_0. Integrating $[\zeta\lambda\mu^2\theta^2\phi\bar{z}\cdot(6)+\zeta\lambda\mu^2\theta^2\phi z\cdot\overline{(6)}]$ on $(-b,b)\times\Omega$, and noting that $z(-b,x)=z(b,x)=0$, we get that

$$
\begin{aligned}
&\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\phi(|\nabla z|^2+|z_{\mathrm{s}}|^2)\mathrm{d}x\mathrm{d}s\leqslant\\
&C[\|\theta z_0\|^2_{L^2(Q)}+\mathrm{e}^{C\lambda}\int_{-b}^{b}\int_{\omega^*}|z|^2\mathrm{d}x\mathrm{d}s].
\end{aligned}
$$

Finally, combining (25) with the above estimate, we obtain the desired estimate (9) immediately. QED.

4 Interpolation inequalities for elliptic equations with Ventcel boundary conditions

In this section, by means of the global Carleman estimate derived in the last section, we present an interpolation inequality for the elliptic equation (6). First, set

$$
\begin{aligned}
&Y=(-1,1)\times\Omega,\ Z^*=(-2,2)\times\Gamma^*,\\
&X^*=(-2,2)\times\omega^*.
\end{aligned}
$$

We have the following interpolation inequality.

Theorem 4 Under the assumption in Theorem 1, there exists a constant $C > 0$, such that for any $\varepsilon > 0$, the solution z of the equation (6) satisfies

$$\|z\|_{H^1(Y)} + \|\nabla_{\mathrm{T}} z\|_{L^2((-1,1)\times\Gamma_1)} \leqslant Ce^{C/\varepsilon}(\|z_0\|_{L^2(Q)}+\|z\|_{L^2(X^*)})+Ce^{-2/\varepsilon}\|z\|_{H^1(Q)}. \tag{26}$$

Proof We borrow some ideas from [29]. Note that there is no boundary condition for z at $s = \pm 2$. Therefore, we introduce a cut-off function $\varphi(\cdot) \in C_0^\infty(-b, b)$ such that

$$\begin{cases} 0 \leqslant \varphi(s) \leqslant 1, & |s| < b, \\ \varphi(s) = 1, & |s| \leqslant b_0. \end{cases} \tag{27}$$

Next, we put $w = \varphi z$. Then, by (6), it follows that

$$\begin{cases} w_{\mathrm{ss}} + \Delta w + id(x)w_{\mathrm{s}} = F_1, & Q, \\ \dfrac{\partial w}{\partial \nu} + p(x)w - \Delta_{\mathrm{T}} w = 0, & \Sigma_1, \\ w = 0, & \Sigma_2, \end{cases} \tag{28}$$

where $F_1 = \varphi_{\mathrm{ss}} z + 2\varphi_{\mathrm{s}} z_{\mathrm{s}} + \varphi z_0 + i\varphi_{\mathrm{s}} d(x)z$.

By applying Theorem 3 to the equation (28), we have

$$\lambda\mu^2 \int_{-b}^{b}\int_\Omega \theta^2\phi(|\nabla w|^2+|w_{\mathrm{s}}|^2+\lambda^2\mu^2\phi^2|w|^2)\mathrm{d}x\mathrm{d}s+ \lambda^2\mu^2\int_{-b}^{b}\int_{\Gamma_1}\theta^2\phi^2|\nabla_{\mathrm{T}} w|^2\mathrm{d}\Gamma\mathrm{d}s \leqslant C[\int_{-b}^{b}\int_\Omega \theta^2|F_1|^2\mathrm{d}x\mathrm{d}s + e^{C\lambda}\int_{-b}^{b}\int_{\omega^*}|w|^2\mathrm{d}x\mathrm{d}s].$$

Recalling the definition of ϕ in (8), we see that

$$\begin{cases} \phi(s,\cdot) \geqslant 2 + e^\mu, & |s| \leqslant 1, \\ \phi(s,\cdot) \leqslant 1 + e^\mu, & b_0 \leqslant |s| \leqslant b. \end{cases} \tag{29}$$

Let $\delta = 2 + e^\mu$. Then by (27) and (29), we get that

$$\lambda e^{2\lambda\delta}\int_{-1}^{1}\int_\Omega(|\nabla z|^2 + |z_{\mathrm{s}}|^2 + |z|^2)\mathrm{d}x\mathrm{d}s + \lambda^2\mu^2\int_{-1}^{1}\int_{\Gamma_1}\theta^2\phi^2|\nabla_{\mathrm{T}} z|^2\mathrm{d}\Gamma\mathrm{d}s \leqslant Ce^{C\lambda}[\int_{-2}^{2}\int_\Omega|z_0|^2\mathrm{d}x\mathrm{d}s + \int_{-2}^{2}\int_{\omega^*}|z|^2\mathrm{d}x\mathrm{d}s] + Ce^{2\lambda(\delta-1)}\int_{(-b,-b_0)\bigcup(b_0,b)}\int_\Omega(|z|^2+|z_{\mathrm{s}}|^2)\mathrm{d}x\mathrm{d}s.$$

By the above estimate, one concludes that there exists an $\varepsilon_1 > 0$, such that for any $\varepsilon \in (0, \varepsilon_1]$, it holds that

$$\|z\|_{H^1(Y)} + \|\nabla_{\mathrm{T}} z\|_{L^2((-1,1)\times\Gamma_1)} \leqslant Ce^{C/\varepsilon}[\|z_0\|_{L^2(Q)} + \|z\|_{L^2(X^*)}] + Ce^{-2/\varepsilon}\|z\|_{H^1(Q)},$$

which yields the desired result. QED.

5 Proof of a resolvent estimate

In this section, we give a proof of our main result.

Proof of Theorem 2 First, for any $\beta \in \mathbb{R}$, $F = (f^0, f^1) \in H$ and $U = (u^0, u^1) \in \mathcal{D}(A)$, it is easy to show that the equation $(A-i\beta I)U = F$ is equivalent to

$$\begin{cases} \Delta u^0 + \beta^2 u^0 - i\beta d(x)u^0 = \\ d(x)f^0 + i\beta f^0 + f^1, & \Omega, \\ \dfrac{\partial u^0}{\partial \nu} + p(x)u^0 - \Delta_{\mathrm{T}} u^0 = 0, & \Gamma_1, \\ u^0 = 0, & \Gamma_2, \\ u^1 = f^0 + i\beta u^0, & \Omega. \end{cases} \tag{30}$$

Put

$$v = e^{-\beta s}u^0. \tag{31}$$

Let $\tilde{f} = d(x)f^0 + i\beta f^0 + f^1$. It is easy to check that v satisfies the following equation:

$$\begin{cases} v_{\mathrm{ss}} + \Delta v + id(x)v_{\mathrm{s}} = \tilde{f}e^{-\beta s}, & \mathbb{R}\times\Omega, \\ \dfrac{\partial v}{\partial \nu} + p(x)v - \Delta_{\mathrm{T}} v = 0, & \mathbb{R}\times\Gamma_1, \\ v = 0, & \mathbb{R}\times\Gamma_2. \end{cases} \tag{32}$$

Now, by (31), we have the following estimates.

$$\begin{cases} \|u^0\|_{H^1(\Omega)} \leqslant Ce^{C|\beta|}\|v\|_{H^1(Y)}, \\ \|v\|_{H^1(Q)} \leqslant Ce^{C|\beta|}\|u^0\|_{H^1(\Omega)}, \\ \|v\|_{L^2(X^*)} \leqslant Ce^{C|\beta|}\|u^0\|_{L^2(\omega^*)}. \end{cases} \tag{33}$$

Applying Theorem 4 to (32), by (33), we have

$$\|u^0\|_{\mathrm{V}} \leqslant Ce^{C|\beta|}(\|f^0\|_{L^2(\Omega)}+\|f^1\|_{L^2(\Omega)}+\|u^0\|_{L^2(\omega^*)}). \tag{34}$$

On the other hand, multiplying (30) by $\bar{u}^0$ and integrating it on Ω, it follows that

$$\int_\Omega[d(x)f^0 + i\beta f^0 + f^1]\bar{u}^0\mathrm{d}x = \beta^2\|u^0\|^2_{L^2(\Omega)} - \int_\Omega|\nabla u^0|^2\mathrm{d}x - \int_{\Gamma_1}[|\nabla_{\mathrm{T}} u^0|^2 + p(x)|u^0|^2]\mathrm{d}\Gamma - i\beta\int_\Omega d(x)|u^0|^2\mathrm{d}x. \tag{35}$$

Taking the imaginary part in both sides of (35) , we conclude that

$$|\beta|\int_{\omega^*}c_0|u^0|^2\mathrm{d}x \leqslant C\|df^0 + i\beta f^0 + f^1\|_{L^2(\Omega)}\|u^0\|_{L^2(\Omega)}. \tag{36}$$

Hence, combining (34) and (36), we have

$$\|u^0\|_{\mathrm{V}} \leqslant Ce^{C|\beta|}(\|f^0\|_{L^2(\Omega)} + \|f^1\|_{L^2(\Omega)}). \tag{37}$$

Recalling that $u^1 = f^0 + i\beta u^0$, it follows that

$$\|u^1\|_{L^2(\Omega)} \leqslant \|f^0\|_{L^2(\Omega)} + |\beta|\|u^0\|_{L^2(\Omega)} \leqslant Ce^{C|\beta|}(\|f^0\|_{L^2(\Omega)} + \|f^1\|_{L^2(\Omega)}). \tag{38}$$

By (37)–(38), we know that $A-i\beta I$ is injective. Therefore, $A - i\beta I$ is bijective from $D(A)$ to H. Moreover,

$$\|(A - i\beta I)^{-1}\|_{\mathcal{L}(H)} \leqslant Ce^{C|\beta|}.$$

QED.

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