一类非线性系统的自适应抗测量噪声的输出反馈镇定

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摘要: 本文研究一类非线性系统的自适应抗测量噪声的输出反馈镇定问题. 所研究的非线性系统输出中存在正的且有界的乘性噪声. 非线性项的增长率为一个未知常数乘以输出的幂函数加上带有时滞输出的幂函数. 首先, 证明一个矩阵不等式. 其次, 设计含有3个时变增益的输出反馈控制器, 并给出增益的自适应律, 然后, 构造适当的Lyapunov-Krasovskii泛函, 给出确保闭环系统渐近稳定的充分条件. 最后, 仿真实验验证该方法的可行性和有效性.

关键词: 乘积形式噪声; 自适应镇定; 非线性系统; 时滞; 输出反馈

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Adaptive anti-measurement-disturbance stabilization fora class of nonlinear systems via output feedback

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Abstract: In this paper, we study adaptive anti-measurement-disturbance stabilization for a class of nonlinear systems via output feedback. In the output of the systems, there exist multiplicative noises which are assumed to be positive and have known upper and lower bounds. The growth rate of the nonlinear terms has an unknown constant multiplied by a power function of the output and a power function of the output with time delay. Firstly, a matrix inequality is developed. Secondly, we design an output feedback stabilizer with three time-varying gains, and give adaptive laws of the gains as well. Then, a Lyapunov-Krasovskii functional is constructed, and sufficient conditions are derived to ensure that the closed-loop system is asymptotically stable. Finally, numerical simulations are provided to verify the feasibility and effectiveness of the design method.

Key words: multiplicative noises; adaptive stabilization; nonlinear system; time-delay; output feedback

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1 Introduction

The problem of stabilization has been well studied for nonlinear systems via output feedback in the past few decades. Most of the results are derived under the condition that the output can be measured precisely [1–4]. However, since the influence of disturbance or sensor error, sometimes, we cannot get the accurate values of the output. For this reason, the synthesis problem has been studied for nonlinear systems with output containing disturbance, such as $y = x_1 + \rho$, where ρ denotes disturbance. For instance, an adaptation-gain observer was designed for a class of nonlinear systems with measurement noises [5]. The authors in [6] addressed an L_1 adaptive output-feedback descriptor for multi-variable nonlinear systems with measurement disturbances.

However, sometimes the error in the output is not related to time, but related to the states. Therefore, researchers proposed the assumption $y = \varphi(x_1)$ [7–9], where $\varphi(\cdot)$ is a function with respect to x_1. In order to stabilize this type of systems, it is usually needed to assume that y is differentiable. For example, the authors in [8] studied output feedback stabilization for uncertain nonlinear systems with unknown growth rate and unknown output function. A design method was proposed to solve the problem of sampled-data output feedback

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stabilization for nonlinear systems with unknown output function [9].

Recently, a new output function error model like $y = \theta(t)x_1$ has been proposed, where $\theta(t)$ is a function with respect to time. Compared with the previous model, it is not necessary to assume that the function y is derivable when considering the stabilization problem for this kind of nonlinear systems. In fact, it is only assumed that $\theta(t)$ is a bounded function [10–12]. The authors in [10] proposed a dual-domination approach to copy with the problem of output-feedback stabilization for nonlinear systems with unknown measurement sensitivity. More specifically, in [11], the authors developed a new stochastic adaptive dual-domination approach to deal with the problem of stabilization for stochastic strict-feedback systems with sensor uncertainty. A large bound of measurement sensitivity was allowed to achieve the regulation of nonlinear systems with unknown growth constant rate [12]. However, in practice, nonlinear systems with time-varying growth rate are usually applied to model the circuits with nonlinear resistance [13–14] and business cycles [15]. Therefore, it is interesting to research the problem of anti-measurement-disturbance output feedback stabilization for a class of nonlinear systems with multiplicative noises and with time-varying, time-delay growth rate.

In this paper, we study the problem of output feedback stabilization for nonlinear systems with unknown measurement sensitivity. The growth rate of the nonlinear terms has an unknown constant multiplied by a power function of the output and a power function of the output with time delay. Firstly, we present a matrix inequality. Then, based on this matrix inequality, an output feedback controller is constructed with three time-varying gains to stabilize the nonlinear system. At last, a Lyapunov-Krasovskii functional is proposed and sufficient conditions are derived to ensure that the closed-loop system is asymptotically stable. Our major contributions include: 1) A useful matrix inequality is proposed. 2) Compared with the results in [16–17], the boundedness of the measurement disturbances $\theta(t)$ is enlarged as $0 < \theta(t) < +\infty$, and the growth rate of the nonlinear terms is time-varying and dependent on the output.

The remainder of this paper is organized as follows. In Section 2, we present some useful lemmas and problem description. In Section 3, an output feedback controller is designed based on a specially constructed observer and three time-varying gains. In Section 4, we give our main results: sufficient conditions are proposed to ensure asymptotical stability of the closed-loop system. Numerical simulations are provided to illustrate the validity of the proposed design methods in Section 5. This paper is concluded in Section 6.

2 Preliminaries and problem description

In this paper, we consider an n-order ($n \geqslant 2$) single-input single-output (SISO) uncertain nonlinear system

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(t, \bar{x}_i),\ i = 1, \cdots, n-1, \\ \dot{x}_n = u + f_n(t, \bar{x}_n), \\ y = \theta(t)x_1, \end{cases} \tag{1}$$

where $\bar{x}_i = (x_1 \ \ \cdots \ \ x_i)^{\mathrm{T}} \in \mathbb{R}^i$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, control input and measurement output, respectively. The sensor sensitivity $\theta(t)$ ($t \in \mathbb{R}^+$) is an unknown continuous function. The functions f_i: $\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous and satisfy the following assumptions.

Assumption 1 [3, 18] There exists a known real number $p \geqslant 0$ and an unknown constant $c > 0$ such that

$$\begin{aligned} &|f_i(t, \bar{x}_i)| \leqslant \\ &c(1 + |x_1|^p) \sum_{j=1}^{i} |x_i(t)| + \\ &c(1 + |x_1(t - \tau(t))|^p) \sum_{j=1}^{i} |x_i(t - \tau(t))|, \end{aligned}$$

where $\tau(t)$ represents time-delay and satisfies that $0 \leqslant \dot{\tau}(t) \leqslant \hat{\tau} < 1$, $\hat{\tau}$ is a known constant.

Remark 1 Different from [11–12], the growth rate is a time-varying function in this paper. When $p = 0$, the time-varying growth rate is reduced to a constant growth rate. Therefore, the constant growth rate can be regarded as a special case of the time-varying growth rate. Moreover, unlike [10, 17], the constant c of the growth rate is unknown. With the introduction of unknown constant, time-delay and sensor sensitivity, it is more difficult to design a stabilizer for the nonlinear system (1).

Remark 2 In practice, the nonlinear system with time-varying growth rate satisfied Assumption 1 is usually applied to model the circuits with nonlinear resistance [13–14] and business cycles [15]. The dynamical equation called the forced van der Pol equation [19–20] is given as follows:

$$\ddot{\vartheta} + \mu(1 - \vartheta^2)\dot{\vartheta} + \vartheta = u, \tag{2}$$

where μ is an unknown constant. The authors in [20] discussed in detail how an actual nonlinear RLC series circuit was transformed into the equation (2).

Suppose that only ϑ is measurable. Under the coordinate transformation $x_1 = \vartheta$, $x_2 = \dot{\vartheta}$, we have

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u - x_1 - \mu(1 - x_1^2)x_2, \\ y = x_1. \end{cases} \tag{3}$$

If let $c = \max\{1, |\mu|\}, p = 2$, then the condition in Assumption 1 holds. Thus, the system (3) has the form of the system (1).

Assumption 2 As in [12], the sensor sensitivity $\theta(t)$ is assumed to be unknown, continuous and bounded. Moreover, there exists positive constants $0 < \theta_l \leqslant 1$ and $1 < \theta_u \leqslant \infty$ such that $\theta_l \leqslant \theta(t) \leqslant \theta_u$, for all $t \geqslant 0$.

Remark 3 In this paper, $\theta(t)$ is assumed to be an unknown continuous function with known upper and lower bounds, but does not need to be derivable. In fact, there always exists a multiplicative noise $\theta(t)$. For instance, in [21], the authors pointed out that the magnetic displacement sensor of the bearing suspension system has a sensor error of $\pm$ 10%, which means $\theta(t)$ is a bounded time-varying function ranging from 0.9 to 1.1. Because of its unique properties, it has been widely studied [10–12, 16].

Compared with [11, 16], in this paper, the allowable measurement error range is enlarged from 0 to $+\infty$. Thus, the proposed method can be applied to nonlinear systems not only with a multiplicative noise $\theta(t)$ close to 1, but also with a multiplicative noise in the interval $(0, +\infty)$.

We also need the following inequalities to derive our main results.

Lemma 1[22] For $(x \ \ y)^{\mathrm{T}} \in \mathbb{R}^2$, the following Young's inequality holds:

$$xy \leqslant \frac{v^p}{p}|x|^p + \frac{1}{qv^q}|y|^q,$$

where $v > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p-1)(q-1) = 1$.

Lemma 2[23] For $p \in [1, +\infty)$ and any $x_i \in \mathbb{R}, i = 1, \cdots, n$, the following inequality holds:

$$(|x_1| + \cdots + |x_n|)^p \leqslant n^{p-1}(|x_1|^p + \cdots + |x_n|^p).$$

Lemma 3[12] Under Assumption 2, let

$$l_1 = b_2 + \frac{1}{2} + l_0,$$

$$l_i = b_i l_{i-1} - b_i \prod_{k=2}^{i} b_k + \prod_{k=2}^{i+1} b_k,\ i = 2, \cdots, N,$$

where l_0 is a positive constant, l_0^* satisfies

$$\rho_1(1-\theta(t)) - \frac{l_0^*\theta(t)}{2} \leqslant 0,$$

and the following inequalities for $i = 2, \cdots, N$,

$$\frac{2}{\kappa(n-1)^2}\left(\frac{l_0^*\theta(t)}{2} - \rho_1(1-\theta(t))\right) - (1-\theta(t))^2\rho_i^2 \geqslant 0,$$

$$\rho_1 = b_2 + \frac{1}{2}, \rho_i = b_i \prod_{k=2}^{i} b_k - \prod_{k=2}^{i+1} b_k,$$

$$b_i = b_{i+1} + \frac{i}{2} + \frac{1}{\kappa} + \bar{b}_i,$$

$i = 2, \cdots, N$, $b_{N+1} = 0$, $\bar{b}_N = 0$, κ is a positive constant, and

$$\bar{b}_i = \frac{1}{2}\sum_{m=i+1}^{N-1}(\bar{b}_m + \frac{1}{\kappa} + \frac{m}{2})^2 \prod_{k=i+1}^{m} b_k^2 + \frac{1}{2}b_N^2 \prod_{k=i+1}^{N} b_k^2,\ i = 2, \cdots, N-1.$$

Let A_{L} be an $N \times N$ matrix and $P_{\mathrm{L}} = P_1^{\mathrm{T}} P_1$ is a positive definite matrix as

$$A_{\mathrm{L}} = \begin{pmatrix} -l_1\theta(t) & 1 & 0 & \cdots & 0 \\ -l_2\theta(t) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -l_{N-1}\theta(t) & 0 & 0 & \cdots & 1 \\ -l_N\theta(t) & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -b_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -b_N & 1 \end{pmatrix}.$$

Then, for $l_0 > l_0^*$, we have the following inequality:

$$A_{\mathrm{L}} P_{\mathrm{L}} + P_{\mathrm{L}} A_{\mathrm{L}} \leqslant -\theta_{\mathrm{M}} I,$$

where $\theta_{\mathrm{M}} = \lambda_{\min}(P_{\mathrm{L}}) \min\{l_0\theta_l, \frac{1}{\kappa}\}$, I is an $N \times N$ identity matrix.

Remark 4 The different between Lemma 3 and Lemma 1 in [12] is that a parameter κ is introduced. But the proof process is similar and is omitted here. This parameter κ can bring flexibility when designing the output feedback stabilizer.

Lemma 4 Suppose that the conditions of Lemma 3 hold. For an $N \times N$ matrix $D = \mathrm{diag}\{\sigma, 1+\sigma, \cdots, N-1+\sigma\}$, there exists an appropriate positive constant σ^*, such that when $\sigma > \sigma^*$, we have the following matrix inequality:

$$DP_{\mathrm{L}} + P_{\mathrm{L}} D \geqslant \pi_1 P_{\mathrm{L}},$$

where $\pi_1 > 0$ is a real constant.

Proof Consider a system $\dot{\eta} = D\eta$ with $\eta = [\eta_1 \ \ \eta_2 \ \ \cdots \ \ \eta_N]^{\mathrm{T}}$. Using a transformation $\xi = P_1\eta$, we have

$$\eta_1 = \xi_1,$$

$$\eta_i = \xi_i + \sum_{j=1}^{i-1}\xi_j \prod_{k=j+1}^{i} b_k,\ i = 2, \cdots, N.$$

Then, that is,

$$\dot{\xi} = P_1 D\eta,$$

$$\dot{\xi}_1 = \sigma\xi_1,$$

$$\dot{\xi}_i = -b_i(i-2+\sigma)(\xi_{i-1} + \sum_{j=1}^{i-2}\xi_j \prod_{k=j+1}^{i-1} b_k) + (i-1+\sigma)(\xi_i + \sum_{j=1}^{i-1}\xi_j \prod_{k=j+1}^{i} b_k) =$$

$$(i-1+\sigma)\xi_i + \sum_{j=1}^{i-1}\xi_j \prod_{k=j+1}^{i} b_k,$$
$$i = 2,\cdots,N.$$

Note that $\xi_i\xi_j \geqslant -\frac{1}{2}\xi_i^2 - \frac{1}{2}\xi_j^2$, and $b_i > 1, i = 2,\cdots,N$. Therefore,

$$\sum_{i=1}^{N}\xi_i\dot{\xi}_i = \sum_{i=1}^{N}(i-1+\sigma)\xi_i^2 + \sum_{i=2}^{N}\xi_i\sum_{j=1}^{i-1}\xi_j\prod_{k=j+1}^{i}b_k \geqslant \sum_{i=1}^{N}(\sigma-\sigma^*)\xi_i^2,$$

where σ^* is an appropriate positive constant and related to b_k. If $\sigma > \sigma^*$, then, we get $\sum_{i=1}^{N}\xi_i\dot{\xi}_i > 0$. Note that

$$\sum_{i=1}^{N}\xi_i\dot{\xi}_i = \frac{1}{2}\frac{\mathrm{d}(\xi^{\mathrm{T}}\xi)}{\mathrm{d}t} = \frac{1}{2}(\dot{\eta}^{\mathrm{T}}P_1^{\mathrm{T}}P_1\eta + \eta^{\mathrm{T}}P_1^{\mathrm{T}}P_1\dot{\eta}) = \frac{1}{2}(\eta^{\mathrm{T}}DP_{\mathrm{L}}\eta + \eta^{\mathrm{T}}P_{\mathrm{L}}D\eta) > 0.$$

Thus, the conclusion holds. □

Remark 5 Note that σ^* increases with the increase of b_k. We can select a larger value of κ to make b_k and σ^* small. For example, when $N = 2$, $\kappa = 10$, we have $b_2 = 1.1$. Choose $\sigma = 0.25$, then $DP_{\mathrm{L}} + P_{\mathrm{L}}D > 0$. If we choose $\kappa = 1$ like [12], when $N = 2$, we have $b_2 = 2$. With the same parameter $\sigma = 0.25$, we get $\lambda_{\min}(DP_{\mathrm{L}} + P_{\mathrm{L}}D) < 0$.

3 Output feedback controller design

In this section, an output feedback controller is constructed for the nonlinear system (1) with unknown sensor sensitivity $\theta(t)$ and the time-varying growth rate shown in Assumption 1.

Firstly, construct the following observer:

$$\begin{cases}\dot{\hat{x}}_i = \hat{x}_{i+1} + l_i(L_1L_3)^i(y-\hat{x}_1),\\ \qquad i = 1,\cdots,n-1,\\ \dot{\hat{x}}_n = u + l_n(L_1L_3)^n(y-\hat{x}_1),\end{cases} \tag{4}$$

where $\hat{x} = (\hat{x}_1 \ \cdots \ \hat{x}_n)^{\mathrm{T}} \in \mathbb{R}^n$ is observer state, the dynamic gains L_1, L_2 and L_3 are updated by

$$\dot{L}_1 = \frac{y^2+\hat{x}_1^2}{1+y^2+\hat{x}_1^2}\Big(\frac{L_2^{2n-3}+1}{(L_1L_2)^{2n-3}L_3^{2\sigma_1}}\Big),\ L_1(0) = 1, \tag{5}$$

$$\dot{L}_2 = \frac{y^2+\hat{x}_1^2}{1+y^2+\hat{x}_1^2}\Big(\frac{1}{(L_1L_2)^{2n-3}L_3^{2\sigma_1}}\Big),\ L_2(0) = 1, \tag{6}$$

and

$$\dot{L}_3 = \max\{-\alpha L_3^2 + \beta L_1(1+(\frac{|y|}{\theta_l})^p)^2, 0\},$$
$$L_3(0) = 1, \tag{7}$$

where

$$\alpha \leqslant \min\{\frac{1}{\pi_1\lambda_{\min}(P_{\mathrm{L}})}, \frac{1}{\pi_2\lambda_{\min}(Q)}\}, \tag{8}$$

and

$$\beta \geqslant \max\{\frac{2-\hat{\tau}}{(1-\hat{\tau})\pi_1\lambda_{\min}(P_{\mathrm{L}})}, \frac{1}{(1-\hat{\tau})\pi_2\lambda_{\min}(Q)}, \frac{1}{(1-\hat{\tau})\pi_1\lambda_{\min}(P_{\mathrm{L}})}\}, \tag{9}$$

and

$$\sigma_1 > \sigma^*, \tag{10}$$

and

$$p < \frac{1}{\sigma_1}. \tag{11}$$

The controller $u(t)$ is given by

$$u = -\sum_{i=1}^{n}(a_i(L_1L_2L_3)^{n-i+1})\hat{x}_i, \tag{12}$$

where $a_i > 0$ $(i = 1,\cdots,n)$ are coefficients of the Hurwitz polynomial $h_1(s) = s^{n+1} + a_1s^n + \cdots + a_ns + a_n$.

Introduce the following change of coordinates:

$$e_i = \frac{x_i - \hat{x}_i}{L_1^{i-1}L_3^{i-1+\sigma_1}},\ i = 1,\cdots,n, \tag{13}$$

$$z_i = \frac{\hat{x}_i}{(L_1L_2)^{i-1}L_3^{i-1+\sigma_1}},\ i = 1,\cdots,n. \tag{14}$$

From (1) (4) (13) and (14), we have

$$\dot{e} = L_1L_3A_{\mathrm{L}}e + (1-\theta(t))L_1L_3Lz_1 + F - \frac{\dot{L}_1}{L_1}D_2e - \frac{\dot{L}_3}{L_3}D_1e, \tag{15}$$

and

$$\dot{z} = L_1L_2L_3Bz + L_1L_3\theta(t)MLe_1 - \frac{\dot{L}_3}{L_3}D_1z - L_1L_3(1-\theta(t))MLz_1 - (\frac{\dot{L}_1}{L_1} + \frac{\dot{L}_2}{L_2})D_2z, \tag{16}$$

where

$e = (e_1 \ \cdots \ e_n)^{\mathrm{T}}$,

$D_1 = \mathrm{diag}\{\sigma_1, 1+\sigma_1, \cdots, n-1+\sigma_1\}$,

$D_2 = \mathrm{diag}\{0, 1, \cdots, n-1\}$,

$L = (l_1 \ l_2 \ \cdots \ l_n)^{\mathrm{T}},\ z = (z_1 \ \cdots \ z_n)^{\mathrm{T}}$,

$$A = \begin{pmatrix} -l_1\theta(t) & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ -l_{n-1}\theta(t) & 0 & \cdots & 1\\ -l_n\theta(t) & 0 & \cdots & 0\end{pmatrix},$$

$$F=\begin{pmatrix}\dfrac{f_1}{L_3^{\sigma_1}}\\ \dfrac{f_2}{L_1L_3^{1+\sigma_1}}\\ \vdots\\ \dfrac{f_n}{L_1^{n-1}L_3^{n-1+\sigma_1}}\end{pmatrix},\ B=\begin{pmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\\ -a_1 & -a_2 & \cdots & -a_n\end{pmatrix},$$

and $M=\operatorname{diag}\{1,\dfrac{1}{L_2},\cdots,\dfrac{1}{L_2^{n-1}}\}$.

Then, by Lemma 1 in [24], there exists a positive definite matrix Q satisfying

$$\begin{aligned}&B^{\mathrm{T}}Q+QB\leqslant -I_n,\\ &D_1Q+QD_1\geqslant \pi_2 Q,\end{aligned}\tag{17}$$

where $\pi_2>0$ is a real constant.

4 Main results

In this section, we construct a Lypunov-Krasovskii functional to derive sufficient conditions to guarantee that the closed-loop system (15)–(16) is asymptotically stable.

Theorem 1 For the system (1) with the Assumptions 1 and 2, if the parameters α, β, σ_1, p satisfy the conditions (8)–(11), then, under the output feedback controller (4)–(7), and (12), the system (1) converges to the equilibrium at origin, which means that $\lim\limits_{t\to+\infty}x(t)=0$, $\lim\limits_{t\to+\infty}\hat{x}(t)=0$.

Proof The derivative of the function $V_1(t)=e^{\mathrm{T}}P_{\mathrm{L}}e$ is given by

$$\begin{aligned}\dot{V}_1(t)\leqslant{}& L_1L_3e^{\mathrm{T}}(A_{\mathrm{L}}^{\mathrm{T}}P_{\mathrm{L}}+P_{\mathrm{L}}A_{\mathrm{L}})e+\\ &2L_1L_3|1-\theta(t)|\|L\|\|P_{\mathrm{L}}\|\|e\|\|z\|+\\ &2\|e\|\|P_{\mathrm{L}}\|\|F\|+2\frac{\dot{L}_1}{L_1}\|D_2\|\|P_{\mathrm{L}}\|\|e\|^2-\\ &\frac{\dot{L}_3}{L_3}e^{\mathrm{T}}(D_1P_{\mathrm{L}}+P_{\mathrm{L}}D_1)e.\end{aligned}\tag{18}$$

From Lemma 1, Assumption 1, (13) and (14), we get

$$\begin{aligned}&\|F\|\leqslant\|F\|_1\leqslant\\ &c(1+|x_1|^p)\sum_{i=1}^{n}\sum_{j=1}^{i}(|e_j|+L_2^{j-1}|z_j|)+\\ &c(1+|x_1(t-\tau(t))|^p)\sum_{i=1}^{n}\sum_{j=1}^{i}(|e_j(t-\tau(t))|+\\ &L_2^{j-1}(t-\tau(t))|z_j(t-\tau(t))|)\leqslant\\ &c(1+|x_1|^p)n\sqrt{n}(\|e\|+L_2^{n-1}\|z\|)+\\ &c(1+|x_1(t-\tau(t))|^p)n\sqrt{n}(\|e(t-\tau(t))\|+\\ &L_2^{n-1}(t-\tau(t))\|z(t-\tau(t))\|).\end{aligned}$$

From (5) and (6), it follows that

$$\dot{L}_1\leqslant 2,\ \dot{L}_2\leqslant 1,$$

and

$$\dot{L}_1=(L_2^{2n-3}\dot{L}_2+\dot{L}_2)\geqslant L_2^{2n-3}\dot{L}_2.\tag{19}$$

Then,

$$\begin{aligned}&L_1-1\geqslant\frac{1}{2(n-1)}(L_2^{2(n-1)}-1),\\ &2(n-1)L_1\geqslant L_2^{2(n-1)}.\end{aligned}$$

Thus,

$$\begin{aligned}&2\|e\|\|P_{\mathrm{L}}\|\|F\|\leqslant\\ &L_1\frac{1}{L_3}(1+|x_1|^p)^2\|e\|^2+\\ &(3+2(n-1))L_3c^2n^3\|P_{\mathrm{L}}\|^2\|e\|^2+\\ &4(n-1)L_3c^2n^3L_1\|P_{\mathrm{L}}\|^2\|z\|^2+\\ &\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2\|e(t-\tau(t))\|^2+\\ &\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2L_1(t-\\ &\tau(t))\|z(t-\tau(t))\|^2.\end{aligned}\tag{20}$$

Note that $(1+(\dfrac{|y|}{\theta_l})^p)^2\geqslant(1+|x_1|^p)^2$, $D_1P_{\mathrm{L}}+P_{\mathrm{L}}D_1\geqslant\pi_1P_{\mathrm{L}}\geqslant\pi_1\lambda_{\min}(P_{\mathrm{L}})I$, $\dot{L}_3\geqslant 0$, and $L_3\geqslant 1$. From Lemma 4 and (7)–(9), we obtain

$$\begin{aligned}&-\frac{\dot{L}_3}{L_3}e^{\mathrm{T}}(D_1P_{\mathrm{L}}+P_{\mathrm{L}}D_1)e\leqslant\\ &\alpha\pi_1\lambda_{\min}(P_{\mathrm{L}})L_3\|e\|^2-\\ &\beta\pi_1\lambda_{\min}(P_{\mathrm{L}})L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|e\|^2\leqslant\\ &L_3\|e\|^2-L_1\frac{1}{L_3}(1+|x_1|^p)^2\|e\|^2-\\ &\frac{1}{1-\hat{\tau}}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|e\|^2.\end{aligned}\tag{21}$$

Substituting (20) and (21) into (18), from Lemma 3, we have

$$\begin{aligned}&\dot{V}_1(t)\leqslant\\ &-\theta_{\mathrm{M}}L_1L_3\|e\|^2+c_1L_1L_3|1-\\ &\theta(t)|\|e\|\|z\|+c_1L_3\|e\|^2+c_1L_1L_3\|z\|^2+\\ &\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2\|e(t-\tau(t))\|^2+\\ &\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2L_1(t-\tau(t))\|z(t-\\ &\tau(t))\|^2-\frac{1}{1-\hat{\tau}}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|e\|^2,\end{aligned}\tag{22}$$

where $c_1=\max\{2\|P_{\mathrm{L}}\|\|L\|,4(n-1)c^2n^3\|P_{\mathrm{L}}\|^2,(3+2(n-1))c^2n^3\|P_{\mathrm{L}}\|^2+1+4\|D_2\|\|P_{\mathrm{L}}\|\}$.

The derivative of $V_2(t)=z^{\mathrm{T}}Qz$ along the system (16) is given as follows:

$$\begin{aligned}\dot V_2(t) \leqslant{}& -L_1L_2L_3\|z\|^2-\frac{\dot L_3}{L_3}z^{\mathrm T}(D_1Q+QD_1)z+\\&2L_1L_3\theta(t)\|M\|\|L\|\|Q\|\|e\|\|z\|+\\&2L_1L_3|1-\theta(t)|\|M\|\|L\|\|Q\|\|z\|^2+\\&2(\frac{\dot L_1}{L_1}+\frac{\dot L_2}{L_2})\|D_2\|\|Q\|\|z\|^2.\end{aligned}$$

Similar to (21), we have

$$\begin{aligned}&-\frac{\dot L_3}{L_3}z^{\mathrm T}(D_1Q+QD_1)z\leqslant\\&L_3\|z\|^2-\frac{1}{1-\hat\tau}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|z\|^2.\end{aligned}$$

Note that $\|ML\|\leqslant\|L\|$. From (17), we have

$$\begin{aligned}\dot V_2(t)\leqslant{}&-L_1L_2L_3\|z\|^2+c_2L_1L_3\theta(t)\|e\|\|z\|+\\&c_2L_1L_3|1-\theta(t)|\|z\|^2+c_2L_3\|z\|^2-\\&\frac{1}{1-\hat\tau}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|z\|^2,\end{aligned}\tag{23}$$

where $c_2=\max\{2\|Q\|\|L\|,1+6\|D_2\|\|Q\|\}$.

Consider the following Lyapunov-Krasovskii functional:

$$V(t)=V_1(t)+V_2(t)+V_3(t)+V_4(t),$$

where

$$V_3(t)=\frac{1}{1-\hat\tau}\frac{1}{L_3}\sum_{i=1}^n\int_{t-\tau(t)}^t\hbar(s)e_i^2(s)\mathrm ds,$$

$$V_4(t)=\frac{1}{1-\hat\tau}\frac{1}{L_3}\sum_{i=1}^n\int_{t-\tau(t)}^t\hbar(s)L_1(s)z_i^2(s)\mathrm ds,$$

and $\hbar(s)=(1+(\frac{|y(s)|}{\theta_l})^p)^2$.

Note that $L_3\geqslant1$, $\frac{1-\dot\tau}{1-\hat\tau}\geqslant1$ and $\dot L_3\geqslant0$. Then,

$$\begin{aligned}&\dot V_3(t)\leqslant\\&\frac{1}{1-\hat\tau}\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|e\|^2-\frac{\dot L_3}{L_3}V_3(t)-\\&\frac{1-\dot\tau}{1-\hat\tau}\frac{1}{L_3}(1+(\frac{|y(t-\tau(t))|}{\theta_l})^p)^2\|e(t-\tau(t))\|^2\leqslant\\&\frac{1}{1-\hat\tau}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|e\|^2-\\&\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2\|e(t-\tau(t))\|^2.\end{aligned}\tag{24}$$

Similar to (24), we have

$$\begin{aligned}\dot V_4(t)\leqslant{}&\frac{1}{1-\hat\tau}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|z\|^2-\\&\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2L_1(t-\\&\tau(t))\|z(t-\tau(t))\|^2.\end{aligned}\tag{25}$$

From (22)–(25), it follows that

$$\begin{aligned}&\dot V(t)=\dot V_1(t)+\dot V_2(t)+\dot V_3(t)+\dot V_4(t)\leqslant\\&-\theta_{\mathrm M}L_1L_3\|e\|^2+c_1L_1L_3|1-\theta(t)|\|e\|\|z\|+\\&c_1L_3\|e\|^2+c_1L_1L_3\|z\|^2-L_1L_2L_3\|z\|^2+\\&c_2L_1L_3\theta(t)\|e\|\|z\|+c_2L_1L_3|1-\\&\theta(t)|\|z\|^2+c_2L_3\|z\|^2.\end{aligned}$$

Then,

$$\begin{aligned}&\dot V(t)\leqslant\\&-\frac{L_1L_3\theta_{\mathrm M}}{2}\|e\|^2-\frac{L_1L_2L_3}{2}\|z\|^2-\\&L_3(\frac{L_1\theta_{\mathrm M}}{4}-c_1)\|e\|^2-\\&L_3(\frac{L_1L_2}{4}-c_1L_1-c_2L_1|1-\theta(t)|-c_2)\|z\|^2-\\&L_3\begin{bmatrix}\|e\|\\\|z\|\end{bmatrix}^{\mathrm T}\begin{bmatrix}\frac{L_1\theta_{\mathrm M}}{4}&\Pi(t)\\\Pi(t)&\frac{L_1L_2}{4}\end{bmatrix}\begin{bmatrix}\|e\|\\\|z\|\end{bmatrix},\end{aligned}\tag{26}$$

where $\Pi(t)=-\frac12(c_1L_1|1-\theta(t)|+c_2L_1\theta(t))$.

Note that $|1-\theta(t)|$, $\theta(t)$ are bounded and c_1, c_2 are two positive constants. The rest proof will be discussed on the following two cases.

Case 1 There exist three positive constants $\hat L_1$, $\hat L_2$ and t^*, such that if $L_1(t)\geqslant\hat L_1$, $L_2(t)\geqslant\hat L_2$, $t\geqslant t^*$, the following conditions hold:

$$\begin{cases}\frac{L_1\theta_{\mathrm M}}{4}\geqslant c_1,\\\frac{L_1L_2}{4}\geqslant c_1L_1+c_2L_1|1-\theta(t)|+c_2,\\\frac{L_1\theta_{\mathrm M}}{4}\frac{L_1L_2}{4}\geqslant\Pi^2(t),\ \forall t\in[t^*,+\infty).\end{cases}\tag{27}$$

Case 2 $L_1(t)\leqslant\hat L_1$ or $L_2(t)\leqslant\hat L_2$, $\forall t\in[0,+\infty)$.

Firstly, we consider the conditions in Case 1 hold. From (26) and (27), it follows that

$$\dot V\leqslant-\frac{L_1L_3\theta_{\mathrm M}}{2}\|e\|^2-\frac{L_1L_2L_3}{2}\|z\|^2.\tag{28}$$

Obviously, we can get $\lim_{t\to+\infty}\|e\|^2=0$ and $\lim_{t\to+\infty}\|z\|^2=0$.

According to (28), we can obtain

$$\dot V\leqslant-c_3(\|e\|^2+\|z\|^2),$$

where c_3 is an appropriate positive constant.

Thus,

$$\begin{aligned}&\int_0^t(\|e\|^2+\|z\|^2)\mathrm dt\leqslant\\&-\frac{1}{c_3}(V(t)-V(0))\leqslant\frac{V(0)}{c_3}<+\infty.\end{aligned}$$

From (6), it follows that

$$\dot L_2=\frac{y^2+\hat x_1^2}{1+y^2+\hat x_1^2}\frac{1}{(L_1L_2)^{2n-3}L_3^{2\sigma_1}}\leqslant$$

$$\frac{y^2+\hat{x}_1^2}{L_3^{2\sigma_1}} \leqslant \theta_u^2(e_1+z_1)^2+z_1^2 \leqslant (2\theta_u^2+1)(e_1^2+z_1^2).$$

Then,

$$L_2-1 \leqslant (2\theta_u^2+1)\int_0^t(\|e\|^2+\|z\|^2)\mathrm{d}t < +\infty,$$

which means that L_2 is bounded.

From (19), we have

$$L_1-1=\frac{1}{2(n-1)}L_2^{2(n-1)}-\frac{1}{2(n-1)}+L_2-1. \tag{29}$$

Note that L_2 is bounded. Thus, L_1 is also bounded. $\lim\limits_{t\to+\infty}e_1(t)=0$ and $\lim\limits_{t\to+\infty}z_1(t)=0$ imply that $|y|=\theta(t)L_3^{\sigma_1}|e_1+z_1|\leqslant C_1L_3^{\sigma_1}$, where C_1 is an appropriate positive constant. Due to $\dot{L}_3\geqslant 0$, $L_3\geqslant 1$, $p\sigma_1<1$ and L_1 is bounded, there exists $t_3>0$ such that

$$\begin{aligned}&-\alpha L_3^2+\beta L_1(1+(\frac{|y|}{\theta_l})^p)^2\leqslant\\&-\alpha L_3^2+C_2L_3^{2p\sigma_1}+2\beta L_1\leqslant 0,\\&\forall t\in[t_3,+\infty),\end{aligned} \tag{30}$$

where C_2 is an appropriate positive constant. Then, we can obtain that $\dot{L}_3=0, \forall t\in[t_3,+\infty)$ and L_3 is bounded. Therefore, we have $\lim\limits_{t\to+\infty}x(t)=0$, $\lim\limits_{t\to+\infty}\hat{x}(t)=0$. Note that L_1, L_2 and L_3 are bounded and $\lim\limits_{t\to+\infty}\hat{x}(t)=0$. From (12), it follows that $\lim\limits_{t\to+\infty}u(t)=0$.

Secondly, we proceed our discussion on Case 2.

From (29), we know that whatever $L_1(t)$ or $L_2(t)$ is bounded, the other is also bounded.

According to (6), we have

$$\infty>L_2-1> \lim_{t\to+\infty}\frac{1}{(L_1(+\infty)L_2(+\infty))^{2n-3}}\int_0^t\frac{y^2+\hat{x}_1^2}{1+y^2+\hat{x}_1^2}\frac{1}{L_3^{2\sigma_1}}\mathrm{d}t.$$

By the Barbalat's Lemma [25], we can get

$$\lim_{t\to+\infty}\frac{x_1^2}{L_3^{2\sigma_1}}=0,\ \lim_{t\to+\infty}\frac{\hat{x}_1^2}{L_3^{2\sigma_1}}=0.$$

Introduce the following change of coordinates:

$$\varepsilon_i=\frac{x_i-\hat{x}_i}{L_1^{*i-1}L_3^{i-1+\sigma_1}},\ i=1,\cdots,n, \tag{31}$$

$$\xi_i=\frac{\hat{x}_i}{(L_1^*L_2^*)^{i-1}L_3^{i-1+\sigma_1}},\ i=1,\cdots,n, \tag{32}$$

where L_1^* and L_2^* are two positive constants satisfying

$$\begin{aligned}L_1^*\geqslant\max\{&L_1(+\infty),\frac{24}{\theta_\mathrm{M}}c^2n^3\|P_\mathrm{L}\|^2L_2^{*2(n-1)},\\&\frac{12}{\theta_\mathrm{M}}(L_2^{*2(n-1)}+3)c^2n^3\|P_\mathrm{L}\|^2,\frac{12}{\theta_\mathrm{M}},\\&\frac{24}{\theta_\mathrm{M}}\|P_\mathrm{L}\|\sqrt{n}\sum_{i=1}^n a_iL_1\},\end{aligned} \tag{33}$$

$$L_2^*\geqslant L_2(+\infty). \tag{34}$$

From (1) (4) (31) and (32), we have

$$\begin{aligned}\dot{\varepsilon}=&L_1^*L_3A_\mathrm{L}\varepsilon+L_1^*L_3\theta(t)L\varepsilon_1-L_1L_3\theta(t)\varGamma L\varepsilon_1-\\&\frac{\dot{L}_3}{L_3}D_1\varepsilon+L_1L_3(1-\theta(t))\varGamma L\xi_1+F^*,\end{aligned} \tag{35}$$

and

$$\begin{aligned}\dot{\xi}=&L_1^*L_2^*L_3A_\mathrm{L}\xi+L_1^*L_2^*L_3\theta(t)L\xi_1+\\&L_1L_3\theta(t)\varGamma EL\varepsilon_1-L_1L_3(1-\theta(t))\varGamma EL\xi_1+\\&D_3\frac{u}{(L_1^*L_2^*)^{n-1}L_3^{n-1+\sigma_1}}-\frac{\dot{L}_3}{L_3}D_1\xi,\end{aligned} \tag{36}$$

where

$$\varepsilon=(\varepsilon_1\ \cdots\ \varepsilon_n)^\mathrm{T},\ \xi=(\xi_1\ \cdots\ \xi_n)^\mathrm{T},$$
$$D_3=(0\ 0\ \cdots\ 1)^\mathrm{T},$$
$$\varGamma=\mathrm{diag}\{1,\frac{L_1}{L_1^*},\cdots,(\frac{L_1}{L_1^*})^{n-1}\},$$
$$E=\mathrm{diag}\{1,\frac{1}{L_2^*},\cdots,(\frac{1}{L_2^*})^{n-1}\},$$
$$F^*=\begin{pmatrix}\dfrac{f_1}{L_3^{\sigma_1}}\\ \dfrac{f_2}{L_1^*L_3^{1+\sigma_1}}\\ \vdots\\ \dfrac{f_n}{L_1^{*n-1}L_3^{n-1+\sigma_1}}\end{pmatrix}.$$

The derivative of the function $V_5(t)=\varepsilon^\mathrm{T}P_\mathrm{L}\varepsilon$ is given by

$$\begin{aligned}&\dot{V}_5(t)\leqslant\\&L_1^*L_3\varepsilon^\mathrm{T}(A_\mathrm{L}^\mathrm{T}P_\mathrm{L}+P_\mathrm{L}A_\mathrm{L})\varepsilon+\\&2L_1^*L_3|\theta(t)|\|L\|\|P_\mathrm{L}\|\|\varepsilon\||\varepsilon_1|+\\&2L_1L_3|\theta(t)|\|\varGamma\|\|L\|\|P_\mathrm{L}\|\|\varepsilon\||\varepsilon_1|+\\&2L_1L_3|1-\theta(t)|\|\varGamma\|\|L\|\|P_\mathrm{L}\|\|\varepsilon\|\|\xi_1\|-\\&\frac{\dot{L}_3}{L_3}\varepsilon^\mathrm{T}(D_1P_\mathrm{L}+P_\mathrm{L}D_1)\varepsilon+2\|P_\mathrm{L}\|\|\varepsilon\|\|F^*\|\leqslant\\&-\frac{7}{12}\theta_\mathrm{M}L^*{}_1L_3\|\varepsilon\|^2+c_4L_1^*L_3|\theta(t)|^2|\varepsilon_1|^2+\\&c_4L_1^*L_2^*L_3|1-\theta(t)|^2|\xi_1|^2+\frac{\theta_\mathrm{M}}{12}L_1^*L_2^*L_3\|\xi\|^2+\\&\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2\|\varepsilon(t-\tau(t))\|^2+\\&\frac{1}{L_3}(1+|x_1(t-\tau(t))|^p)^2L_1(t-\tau(t))\|\xi(t-\\&\tau(t))\|^2-\frac{1}{1-\hat{\tau}}L_1\frac{1}{L_3}(1+(\frac{|y|}{\theta_l})^p)^2\|\varepsilon\|^2,\end{aligned} \tag{37}$$

where $c_4 = \dfrac{24}{\theta_{\mathrm{M}}}\|L\|^2\|P_{\mathrm{L}}\|^2$.

Calculate the derivative of the function $V_6(t) = \xi^{\mathrm{T}} P_{\mathrm{L}} \xi$. Then,

$$\begin{aligned}
&\dot{V}_6(t) \leqslant \\
&L_1^* L_2^* L_3 \xi^{\mathrm{T}}(A_{\mathrm{L}}^{\mathrm{T}} P_{\mathrm{L}} + P_{\mathrm{L}} A_{\mathrm{L}})\xi + \\
&2L_1^* L_2^* L_3 |\theta(t)| \|L\| \|P_{\mathrm{L}}\| \|\xi\| |\xi_1| + \\
&2L_1^* L_3 |\theta(t)| \|\varGamma\| \|E\| \|L\| \|P_{\mathrm{L}}\| \|\xi\| |\varepsilon_1| + \\
&2L_1^* L_3 |1-\theta(t)| \|\varGamma\| \|E\| \|L\| \|P_{\mathrm{L}}\| \|\xi\| \|\xi_1\| + \\
&2\xi^{\mathrm{T}} P_{\mathrm{L}} D_3 \frac{u}{(L_1^* L_2^*)^{n-1} L_3^{n-1+\sigma_1}} - \\
&\frac{\dot{L}_3}{L_3}\xi^{\mathrm{T}}(D_1 P_{\mathrm{L}} + P_{\mathrm{L}} D_1)\xi \leqslant \\
&-\frac{7}{12}\theta_{\mathrm{M}} L_1^* L_2^* L_3 \|\xi\|^2 + c_4 L_1^* L_2^* L_3 |\theta(t)|^2 |\xi_1|^2 + \\
&c_4 L_1^* L_2^* L_3 |\theta(t)|^2 |\varepsilon_1|^2 + \\
&c_4 L_1^* L_2^* L_3 |1-\theta(t)|^2 |\xi_1|^2 - \\
&\frac{1}{1-\hat{\tau}} L_1 \frac{1}{L_3}(1 + (\frac{|y|}{\theta_l})^p)^2 \|\xi\|^2. \qquad (38)
\end{aligned}$$

Consider the following Lyapunov-Krasovskii functional:

$$V_9(t) = V_5(t) + V_6(t) + V_7(t) + V_8(t),$$

where

$$V_7(t) = \frac{1}{1-\hat{\tau}}\frac{1}{L_3}\sum_{i=1}^{n}\int_{t-\tau(t)}^{t} \hbar(s)\varepsilon_i^2(s)\mathrm{d}s,$$

$$V_8(t) = \frac{1}{1-\hat{\tau}}\frac{1}{L_3}\sum_{i=1}^{n}\int_{t-\tau(t)}^{t} \hbar(s)L_1(s)\xi_i^2(s)\mathrm{d}s.$$

Note that $L_3 \geqslant 1$, $\dfrac{1-\dot{\tau}}{1-\hat{\tau}} \geqslant 1$ and $\dot{L}_3 \geqslant 0$. Similar to (24), it follows that

$$\begin{aligned}
\dot{V}_7(t) \leqslant\ & \frac{1}{1-\hat{\tau}} L_1 \frac{1}{L_3}(1 + (\frac{|y|}{\theta_l})^p)^2 \|\varepsilon\|^2 - \\
& \frac{1}{L_3}(1 + |x_1(t-\tau(t))|^p)^2 \|\varepsilon(t-\tau(t))\|^2. \qquad (39)
\end{aligned}$$

Similar to (39), we have

$$\begin{aligned}
\dot{V}_8(t) \leqslant\ & \frac{1}{1-\hat{\tau}} L_1 \frac{1}{L_3}(1 + (\frac{|y|}{\theta_l})^p)^2 \|\xi\|^2 - \\
& \frac{1}{L_3}(1 + |x_1(t-\tau(t))|^p)^2 L_1(t- \\
& \tau(t)) \|\xi(t-\tau(t))\|^2. \qquad (40)
\end{aligned}$$

Based on (37)–(40), we obtain

$$\begin{aligned}
&\dot{V}_9(t) = \\
&\dot{V}_5(t) + \dot{V}_6(t) + \dot{V}_7(t) + \dot{V}_8(t) \leqslant \\
&-\frac{7}{12}\theta_{\mathrm{M}} L_1^* L_3 \|\varepsilon\|^2 + c_4 L_1^* L_3 |\theta(t)|^2 |\varepsilon_1|^2 + \\
&c_4 L_1^* L_2^* L_3 |1-\theta(t)|^2 |\xi_1|^2 + \frac{\theta_{\mathrm{M}}}{12} L_1^* L_2^* L_3 \|\xi\|^2 - \\
&\frac{7}{12}\theta_{\mathrm{M}} L_1^* L_2^* L_3 \|\xi\|^2 + c_4 L_1^* L_2^* L_3 |\theta(t)|^2 |\xi_1|^2 + \\
&c_4 L_1^* L_2^* L_3 |\theta(t)|^2 |\varepsilon_1|^2 + \\
&c_4 L_1^* L_2^* L_3 |1-\theta(t)|^2 |\xi_1|^2 \leqslant \\
&-\frac{1}{3}\theta_{\mathrm{M}} L_1^* L_3 \|\varepsilon\|^2 - \frac{1}{3}\theta_{\mathrm{M}} L^*{}_1 L_2^* L_3 \|\xi\|^2 - \\
&L_1^* L_3 (\frac{\theta_{\mathrm{M}}}{4}\|\varepsilon\|^2 - c_4 |\theta(t)|^2 |\varepsilon_1|^2 - \\
&c_4 L_2^* |\theta(t)|^2 |\varepsilon_1|^2) - L_1^* L_2^* L_3 (\frac{\theta_{\mathrm{M}}}{6}\|\xi\|^2 - \\
&2c_4 |1-\theta(t)|^2 |\xi_1|^2 - c_4 |\theta(t)|^2 |\xi_1|^2). \qquad (41)
\end{aligned}$$

Note that $|\theta(t)|$, $|1-\theta(t)|$ are bounded. There exist appropriate positive constants C_3, C_4, C_5 such that (41) can be rewritten as

$$\begin{aligned}
&\dot{V}_9(t) \leqslant \\
&-C_3 L_3 (\|\varepsilon\|^2 + \|\xi\|^2) - \\
&L_3 (C_4 (\|\varepsilon\|^2 + \|\xi\|^2) - C_5 (|\varepsilon_1|^2 + |\xi_1|^2)).
\end{aligned}$$

Thus, if $\|\varepsilon\|^2 + \|\xi\|^2 \geqslant \dfrac{C_5}{C_4}(|\varepsilon_1|^2 + |\xi_1|^2)$, we have $\dot{V}_9 \leqslant 0$. $\|\varepsilon\|^2 + \|\xi\|^2$ is ultimately bounded by $\dfrac{C_5}{C_4}(|\varepsilon_1|^2 + |\xi_1|^2)$. Due to

$$\lim_{t\to+\infty}\frac{x_1^2}{L_3^{2\sigma_1}} = \lim_{t\to+\infty}\frac{\hat{x}_1^2}{L_3^{2\sigma_1}} = 0,$$

we have $\lim\limits_{t\to+\infty}|\varepsilon_1|^2 = 0$, $\lim\limits_{t\to+\infty}|\xi_1|^2 = 0$. It is obvious that the ultimate bound of $\|\varepsilon\|^2 + \|\xi\|^2$ becomes to 0 as $t \to +\infty$.

Therefore, we have $\lim\limits_{t\to+\infty}\|\varepsilon\| = 0$, $\lim\limits_{t\to+\infty}\|\xi\| = 0$. Similar to (30), we known that L_3 is bounded. Then, $\lim\limits_{t\to+\infty}x(t) = \lim\limits_{t\to+\infty}\hat{x}(t) = 0$. Note that L_1, L_2 and L_3 are bounded and $\lim\limits_{t\to+\infty}\hat{x}(t) = 0$. According to (12), it follows that $\lim\limits_{t\to+\infty}u(t) = 0$. □

5 Numerical simulations

In this section, we use two simulation examples to demonstrate the effectiveness of our adaptive anti-measurement-disturbance controller design for nonlinear systems with time-varying, time-delay growth rate. In addition, the third example is applied to compare the performance of our method with the method proposed in [12].

Example 1 Consider the following SISO nonlinear system (3) with sensor uncertainty:

$$\begin{cases}\dot{x}_1 = x_2, \\ \dot{x}_2 = u - x_1 - \mu(1-x_1^2)x_2, \\ y = \theta(t)x_1,\end{cases} \qquad (42)$$

where μ is an unknown constant. In this example, we select $\theta(t) = 1 + 0.5\sin t$ and $\theta(t) = 1.4 + \sin t$. It is obvious that system (42) satisfies Assumption 1 with $p = 2$, $c = \max\{1, |\mu|\}$, $\tau(t) = 0$ and Assumption 2 with $\theta_l = 0.4$, $\theta_u = 2.6$. Compared with the examples in literatures [10] and [11], the growth rate of our nonlinear system is no longer a known or an unknown constant, but a time-varying function related to the output. Meanwhile, the range of $\theta(t)$ is not in the vicinity of 1 as in [11], but has been greatly enlarged.

According to Lemma 3, we choose $\kappa = 10$. Thus, $b_2 = 1.1$, $\rho_1 = 1.6$, $\rho_2 = 1.21$. Then, let $l_0 = 150$, $l_1 = 151.6$ and $l_2 = 165.55$. Based on Theorem 1, set $a_1 = 4$, $a_2 = 4$, $\sigma_1 = 0.45$. From (17), we get

$$Q = \begin{bmatrix} 1.125 & 0.125 \\ 0.125 & 0.1563 \end{bmatrix}.$$

Then, $\pi_1 = 0.4$, $\pi_2 = 0.8$, $\lambda_{\min}(P_{\mathrm{L}}) = 0.3496$, $\lambda_{\min}(Q) = 0.1404$, $\hat{\tau} = 0$. According to (8) and (9), we choose $\alpha = 7$, $\beta = 30$. Construct the following controller for the system (42):

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + 151.6L_1L_3(y - \hat{x}_1), \\ \dot{\hat{x}}_2 = u + 165.55(L_1L_3)^2(y - \hat{x}_1), \\ u = -4(L_1L_2L_3)^2\hat{x}_1 - 4L_1L_2L_3\hat{x}_2, \\ \dot{L}_1 = \dfrac{y^2 + \hat{x}_1^2}{1 + y^2 + \hat{x}_1^2}\dfrac{L_2 + 1}{L_1L_2L_3^{0.9}},\ L_1(0) = 1, \\ \dot{L}_2 = \dfrac{y^2 + \hat{x}_1^2}{1 + y^2 + \hat{x}_1^2}\dfrac{1}{L_1L_2L_3^{0.9}},\ L_2(0) = 1, \\ \dot{L}_3 = \max\{-7L_3^2 + 30L_1(1 + (\dfrac{y^2}{0.4^2}))^2, 0\}, \\ L_3(0) = 1. \end{cases} \tag{43}$$

The initial conditions are given as $x_1(0) = 1$, $x_2(0) = 2$, $\hat{x}_1(0) = 2$, $\hat{x}_2(0) = 1$, $L_1(0) = 1$, $L_2(0) = 1$, $L_3(0) = 1$ and the parameter $\mu = 3$. The simulation results are shown in Fig. 1, which verifies that the proposed method is correct and effective.

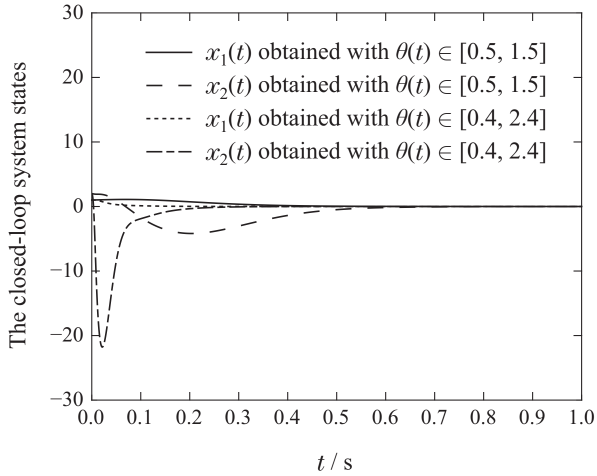

Fig. 1 The trajectories of the states of the closed-loop system with different measurement disturbance

Example 2 In order to verify that our method is still effective in the presence of time delay, we consider a two-stage chemical reactor system [26] as follows:

$$\begin{cases} \dot{x}_1 = \dfrac{1 - R_\beta}{V_\alpha}x_2 - \dfrac{1}{C_\alpha}x_1 - K_\alpha x_1, \\ \dot{x}_2 = \dfrac{E_\alpha}{V_\beta}u - \dfrac{1}{C_\beta}x_2 - K_\beta x_2 + \\ \qquad \dfrac{R_\alpha}{V_\beta}x_1(t - \tau(t)) + \dfrac{R_\beta}{V_\beta}x_2(t - \tau(t)), \\ y = \theta(t)x_1, \end{cases} \tag{44}$$

where x_1 and x_2 are the compositions, u and y are the input and output, R_α and R_β are the recycle flow rates, C_α and C_β are the reactor residence times, E_α is the feed rate, V_α and V_β are the reactor volumes, K_α and K_β are the reaction functions. So as to facilitate the simulation, we choose the following parameters as $R_\alpha = R_\beta = 0.5$, $K_\alpha = K_\beta = 0.5$, $V_\alpha = V_\beta = 0.5$, $C_\alpha = C_\beta = 2$, $E_\alpha = 0.5$. Then, the system (44) can be transformed into

$$\begin{cases} \dot{x}_1 = x_2 - x_1, \\ \dot{x}_2 = u - x_2 + x_1(t - \tau(t)) + x_2(t - \tau(t)), \\ y = \theta(t)x_1. \end{cases} \tag{45}$$

In this example, we choose a non-directed $\theta(t) = 0.4 + 1.6|\cos t|$. It is easy to verify that system (45) satisfies Assumption 1 with $p = 2$, $c = 1$ and Assumption 2 with $\theta_l = 0.4$, $\theta_u = 2$. According to Lemma 3, we choose $\kappa = 10$, thus, $b_2 = 1.1$, $\rho_1 = 1.6$, $\rho_2 = 1.21$. Then, let $l_0 = 40$, $l_1 = 41.6$ and $l_2 = 44.55$.

Based on Theorem 1, set $a_1 = 4$, $a_2 = 4$, $\sigma_1 = 0.45$, $\tau(t) = 0.8$. From (17), we get

$$Q = \begin{bmatrix} 1.125 & 0.125 \\ 0.125 & 0.1563 \end{bmatrix}.$$

Then, $\pi_1 = 0.4$, $\pi_2 = 0.8$, $\lambda_{\min}(P_{\mathrm{L}}) = 0.3496$, $\lambda_{\min}(Q) = 0.1404$, $\hat{\tau} = 0$. From (8) and (9), we choose $\alpha = 7$, $\beta = 20$. Construct the following controller:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + 41.6L_1L_3(y - \hat{x}_1), \\ \dot{\hat{x}}_2 = u + 44.55(L_1L_3)^2(y - \hat{x}_1), \\ u = -4(L_1L_2L_3)^2\hat{x}_1 - 4L_1L_2L_3\hat{x}_2, \\ \dot{L}_1 = \dfrac{y^2 + \hat{x}_1^2}{1 + y^2 + \hat{x}_1^2}\dfrac{L_2 + 1}{L_1L_2L_3^{0.9}},\ L_1(0) = 1, \\ \dot{L}_2 = \dfrac{y^2 + \hat{x}_1^2}{1 + y^2 + \hat{x}_1^2}\dfrac{1}{L_1L_2L_3^{0.9}},\ L_2(0) = 1, \\ \dot{L}_3 = \max\{-7L_3^2 + 20L_1(1 + \dfrac{y^2}{0.4^2})^2, 0\}, \\ L_3(0) = 1. \end{cases} \tag{46}$$

The initial conditions are given as $x_1(0) = 1$, $x_2(0) = 1$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, $L_1(0) = 1$, $L_2(0) = 1$, $L_3(0) = 1$ and the parameter $\mu = 3$. The simulation results are shown in Fig. 2. Obviously, our proposed method is also effective for nonlinear systems with time-delay.

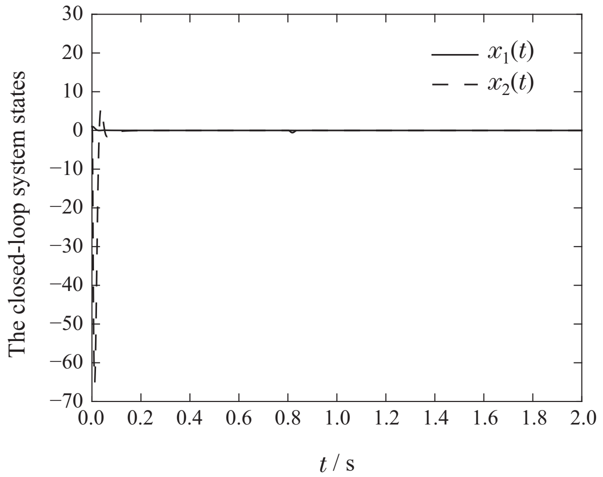

Fig. 2 The trajectories of the states of the closed-loop system

Example 3 In order to compare the effectiveness of our method with the method proposed in [12], we consider the following nonlinear system [12]:

$$\begin{cases} \dot{x}_1 = x_2 + d_1 \sin x_1, \\ \dot{x}_2 = u + d_2 \ln(1 + x_1^2), \\ y = (1.5 + 1.1 \sin t) x_1, \end{cases} \tag{47}$$

where d_1, d_2 are two unknown bounded time-varying functions and $\theta(t) = 1.5 + 1.1 \sin t$. As in [12], the following output feedback controller is constructed:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + 151.5 L_1 (y - \hat{x}_1), \\ \dot{\hat{x}}_2 = u + 299 L_1^2 (y - \hat{x}_1), \\ u = -4(L_1 L_2)^2 \hat{x}_1 - 4 L_1 L_2 \hat{x}_2, \\ \dot{L}_1 = \dfrac{|y| + |\hat{x}_1| + |\hat{x}_2|}{1 + |y| + |\hat{x}_1| + |\hat{x}_2|} \dfrac{L_2 + 1}{L_1 L_2}, L_1(0) = 1, \\ \dot{L}_2 = \dfrac{|y| + |\hat{x}_1| + |\hat{x}_2|}{1 + |y| + |\hat{x}_1| + |\hat{x}_2|} \dfrac{1}{L_1 L_2}, L_2(0) = 1. \end{cases} \tag{48}$$

The initial conditions are given as $x_1(0) = 1$, $x_2(0) = 2$, $\hat{x}_1(0) = 2$, $\hat{x}_2(0) = 1$, $L_1(0) = 1$, $L_2(0) = 1$, $L_3(0) = 1$ and the parameters $d_1 = 1 + \cos t$, $d_2 = 2 - \sin(20t)$. The simulation results are shown in Fig. 3. It can be seen that the system (47) under our presented output feedback controller has a faster convergent speed than that under the controller (48).

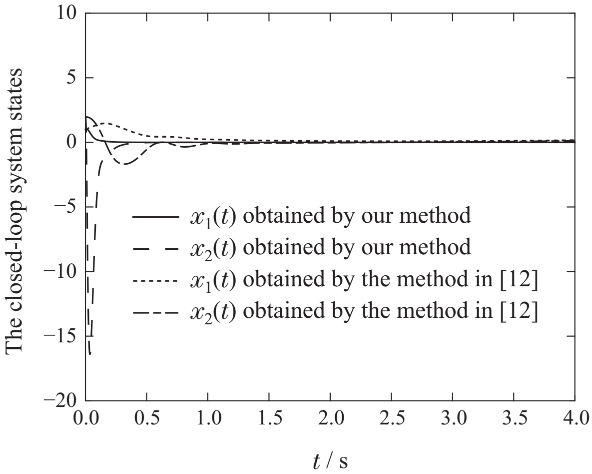

Fig. 3 The trajectories of the states of the closed-loop system with different methods

6 Conclusion

In this paper, we studied anti-measurement-disturbance stabilization for a class of nonlinear systems with unknown growth rate, unknown measurement uncertainty, and time-delay. First, a useful matrix inequality was developed. Then, by using three time-varying gains, an output feedback controller was designed to stabilize the nonlinear system. Based on the obtained matrix inequality and a specially constructed Lyapunov-Krasovskii functional, we derived sufficient conditions to ensure the closed-loop system was asymptotically stable. Finally, numerical simulations were applied to verify the correctness of our theoretic results.

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