第 40 卷第 3 期
2023 年 3 月

控 制 理 论 与 应 用
Control Theory & Applications

Vol. 40 No. 3
Mar. 2023

基于分段多项式李雅普诺夫函数的模糊系统的稳定性分析

吴雨棠, 李丽珍†, 胡德时

(上海电力大学 数理学院, 上海 200090)

摘要: 本文采用最大型分段多项式李雅普诺夫函数研究了多项式模糊系统的闭环稳定性问题. 首先, 本文设计了与分段李雅普诺夫函数对应的切换模糊控制器, 提出了多项式模糊模型稳定的平方和条件, 同时证明了最大型分段多项式李雅普诺夫函数在函数切换点的稳定性. 然后, 设计了相应的路径跟踪优化算法, 对本文非凸的稳定条件进行迭代求解. 最后, 通过两个算例进行仿真与比较, 说明并验证了本文所提出结论的可行有效性.

关键词: 分段多项式李雅普诺夫函数; 平方和方法; 多项式模糊模型; 路径跟踪算法

引用格式: 吴雨棠, 李丽珍, 胡德时. 基于分段多项式李雅普诺夫函数的模糊系统的稳定性分析. 控制理论与应用, 2023, 40(3): 558 – 564

DOI: 10.7641/CTA.2022.10835

Stability analysis of fuzzy systems using piecewise polynomial Lyapunov function

WU Yu-tang, LI Li-zhen†, HU De-shi

(College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China)

Abstract: This paper is devoted to studying the closed-loop stability of polynomial fuzzy systems by using piecewise polynomial Lyapunov function. Firstly, this paper designs the switching fuzzy controller corresponding to the piecewise Lyapunov function, puts forward the sum-of-square conditions for the stability of the polynomial fuzzy model. Meanwhile, the stability of the maximum-type piecewise polynomial Lyapunov function at the switching point is proved. Then, the appropriate path-following optimization algorithm is adopted and designed to solve the nonconvex stability conditions iteratively. Finally, the simulation of two examples and the comparison vertify that the proposed conclusions are feasible and effective.

Key words: piecewise polynomial Lyapunov function; sum of square; polynomial fuzzy model; path-following algorithm

Citation: WU Yutang, LI Lizhen, HU Deshi. Stability analysis of fuzzy systems using piecewise polynomial Lyapunov function. *Control Theory & Applications*, 2023, 40(3): 558 – 564

1 引言

自从1985年T-S模糊模型建立[1], 模糊系统的稳定性以及模糊逻辑控制就成为研究的热点, 得到了许多基于线性矩阵不等式(linear matrix inequality, LMI)的成果, 如文献[2–4]等, 随着研究的深入, T-S模糊模型相关的许多工作都开始研究求解多模糊求和等非凸问题[5–7], 随着T-S模糊系统逐渐发展至多项式模糊系统[8–10], 平方和(sum of square, SOS)方法也被应用其中[8–11], 平方和方法可得到比线性矩阵不等式方法更宽松的稳定结果[8–9, 12].

多项式模糊模型允许系统矩阵中含有多项式, 为研究该模型的稳定性, 常选取含多项式矩阵的李雅普诺夫函数. 若系统矩阵与李雅普诺夫函数中都带有多项式, 那么就不能直接用常见的基于线性矩阵不等式的求解方法了. 所以本文采用平方和方法求解这类多项式问题, 常使用MATLAB的第三方工具箱SOSTOOLS[13]或SOSOPT[14]来进行求解工作.

本文所提出的多项式模糊稳定条件涉及非凸问题, 在诸多涉及求解非凸稳定条件的工作中, 一种典型的做法是将非凸问题转化为凸优化问题求解, 如文献[8–9, 12, 15–17]. 但转化的过程也会给解带来一定的保守性. 所以, 有大量的工作着手研究其他求解非凸

收稿日期: 2021−09−03; 录用日期: 2022−05−17.
†通信作者. E-mail: lilizhen@shiep.edu.cn; Tel.: +86 18116349250.
本文责任编委:徐胜元.
国家自然科学基金项目(61903244)资助.
Supported by the National Natural Science Foundation of China (61903244).

稳定条件的方法, 如路径跟踪算法、粒子群算法、两步法等[18–25]. 本文采用的是用路径跟踪算法进行迭代求解. 路径跟踪算法是一种类梯度算法, 常用来求解含非凸条件的优化问题, 算法的原理在文中稳定性分析部分进行了解释说明. 已经有若干工作证明了路径跟踪算法[26]在求解非凸平方和问题上的有效性[27–30].

本文采用的最大型分段多项式李雅普诺夫函数是分段多项式李雅普诺夫函数的一种. 从函数形式及定义可见, 分段多项式李雅普诺夫函数是普通李雅普诺夫函数的一种推广, 因此分段多项式李雅普诺夫函数能提供更宽松的解[28]. 在文献[27]等工作中, 未能证明最大型分段多项式李雅普诺夫函数在函数切换点的稳定性, 但是在本文第3部分, 给出了最大型分段多项式李雅普诺夫函数稳定性的证明, 这是本文取得的成果之一. 并且通过已有成果[27–28,31–32]可以发现基于最大型分段李雅普诺夫函数提出的稳定条件也更为宽松, 所以本文采用最大型分段多项式李雅普诺夫函数展开研究. 另外, 现有成果中大部分采用分段多项式李雅普诺夫函数的稳定条件都是针对开环系统[28,31,33], 本文以此为切入点进行闭环系统稳定性的研究. 本文主要贡献如下:

1) 在多项式模糊闭环系统中应用最大型分段多项式李雅普诺夫函数, 配合采用切换模糊控制器, 提出了可行的平方和稳定条件. 并设计了相应的路径跟踪优化算法求解. 将分段多项式李雅普诺夫函数应用于多项式模糊闭环模型的稳定性研究中;

2) 证明了最大型分段多项式李雅普诺夫函数在函数切换点上的稳定性. 在理论上论证了采用最大型分段多项式李雅普诺夫函数提出稳定条件的可行性. 并通过两个算例证明了本文方法的有效性.

2　问题描述及预备知识

引理 1 (多项式S-Procedure[34])　对若干多项式 $g_i(x), i=0,1,2,\cdots,N$, 考虑以下两个条件:

S1: 多项式$g_0(x)\leqslant 0$;

S2: 存在若干半正定多项式$h_i(x), i=0,1,2,\cdots,N$, 使得$h_i(x)g_i(x)\leqslant 0, i\neq 0$.

当有$g_0(x)-\sum h_i(x)g_i(x)\leqslant 0, i\neq 0$成立时, 可由S2的成立推出S1.

定义 1　一个多元多项式$p(x_1,x_2,\cdots,x_n)\triangleq p(x)$是SOS. 即存在若干多项式$f_1(x),\cdots,f_m(x)$满足$p(x)=\sum\limits_{i=1}^{m}f_i^2(x)$. 由定义可知若$g(x)$是SOS, 说明$\forall x\in\mathbb{R}^n, g(x)\geqslant 0$.

考虑如下非线性动力系统:

$$\dot{x}(t)=f(x(t))+g(x(t))u(t). \tag{1}$$

其中: $f(x(t))$和$g(x(t))$为光滑非线性函数且满足$f(0)=0$. $x(t)=[x_1(t)\ \ x_2(t)\ \ \cdots\ \ x_n(t)]^{\mathrm{T}}$, $u(t)=[u_1(t)\ \ u_2(t)\ \ \cdots\ \ u_n(t)]^{\mathrm{T}}$分别为状态向量与输入向量. 通过运用扇形非线性理念[35], 非线性系统可建立为多项式模型, 系统(1)可表示为如下多项式模型:

模型规则i:

若$z_1(t)$为M_{i1}, 且$z_p(t)$为M_{ip}, 则

$$\begin{aligned}\dot{\hat{x}}(t)&=A_i(x(t))\hat{x}(x(t))+B_i(x(t))u(t),\\ i&=1,2,\cdots,r,\end{aligned} \tag{2}$$

其中: r为模糊规则数, $z_p(t)$为前件变量, 前件变量可测且与模糊规则数量独立. $\hat{x}(x(t))$为$n\times 1$的列向量, 由$x(t)$中的单项式构成, $x(t)$中的单项式是形如x_1^{a1}, $x_2^{a2},\cdots,x_n^{an}$的函数, 其中$a1,a2,\cdots,an$为非负整数. $A_l(x(t))$, $B_l(x(t))$为多项式系统矩阵.

本文设定: 当且仅当$x(t)=0$时$\hat{x}(x(t))=0$.

模型(2)的解模糊化过程可表示为

$$\dot{x}(t)=\sum_{i=1}^{r}h_i(z(t))A_i(x(t))\hat{x}(x(t))+B_i(x(t))u(t). \tag{3}$$

其中: $h_i\left(z(t)\right)$为隶属度函数,

$$\begin{cases}z(t)=[z_1(t)\ \ z_2(t)\ \ \cdots\ \ z_p(t)],\\ h_i(z(t))=\dfrac{\prod\limits_{k=1}^{p}M_{ij}(z_j(t))}{\sum\limits_{k=1}^{r}\prod\limits_{k=1}^{p}M_{kj}(z_j(t))},\\ \forall i\in[1,r],\ h_i(z(t))\in[0,1],\ \sum\limits_{i=1}^{r}h_i(z(t))=1.\end{cases} \tag{4}$$

为了表示的简洁方便, 下文将省略时间相关记号. 如, 本文将分别使用x, $\hat{x}(x)$, $h_i(z)$表示$x(t)$, $\hat{x}(x(t))$, $h_i(z(t))$.

参考在文献[27,36]中提及的切换模糊控制器, 本文使用如下控制器: 当$V(x)=V_l(x)$时, 控制器切换规则如下:

若$z_1(t)$为M_{i1}, 且$z_p(t)$为M_{ip}, 则

$$u(t)=-F_{il}(x)\hat{x}(x),\ i=1,2,\cdots,r. \tag{5}$$

所以控制器的解模糊化可以表示为

$$u(t)=-\sum_{i=1}^{r}h_iF_{il}(x)\hat{x}(x), \tag{6}$$

其中: l对应此时的$V_l(x)$(当$V(x)=V_l(x)$时, $u(t)=-\sum\limits_{i=1}^{r}h_i(z)F_{i1}(x)\hat{x}(x)$, 当$V_m(x)=V_n(x)$时, $l=\max\{m,n\}$, 具体证明在下文给出). 通过将式(6)代入式(3)可以表示闭环系统

$$\begin{aligned}\dot{x}(t)&=\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j(A_i(x)-B_i(x)F_{jl}(x))\hat{x}(x),\\ &i=1,2,\cdots,r.\end{aligned} \tag{7}$$

本文选用最大型分段多项式李雅普诺夫函数, 来对多项式模型进行稳定性分析. 具体形式如下:

$$\text{Maximum type PPLE}\ V(x)=\max_{1\leqslant l\leqslant N}V_l(x). \tag{8}$$

3 系统稳定分析

在本部分中, 本文将给出上述模型的SOS稳定性条件, 并对所给条件进行分析证明.

定理 1 若存在若干多项式函数$V_l(x)$、多项式$\lambda_{ils}(x)$、多项式矩阵$F_{jl}(x)$以及负标量α满足下列条件, 则系统稳定:

$$V_l(x)-\epsilon(x)\text{是SOS},\ l\in\{1,\cdots,N\}, \tag{9}$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j(\frac{\partial V_l(x)}{\partial x}^{\mathrm T}(A_i(x)-B_i(x)F_{jl}(x))\hat{x}(x)+\alpha V_l(x)-\sum_{s=1}^{k}\lambda_{ils}(x)(V_l(x)-V_s(x)))\text{是 SOS},\ l\in\{1,\cdots,N\}, \tag{10}$$

$$\lambda_{ils}(x)\text{是SOS}\ l,s\in\{1,\cdots,N\},\ i\in\{1,\cdots,r\}, \tag{11}$$

其中$\epsilon(x)$为径向无界的正定多项式.

证 $V_l(x)$为分段李雅普诺夫函数的候选函数, 由李雅普诺夫稳定条件, 为了保证李雅普诺夫函数的正定型, 存在小的径向无界的正定多项式$\epsilon(x)$满足下式:

$$V_l(x)-\epsilon(x)\geqslant 0. \tag{12}$$

李雅普诺夫函数时间导数的闭环路径有如下形式:

$$\dot{V}(x)=(\frac{\partial V(x)}{\partial x})^{\mathrm T}\dot{x}. \tag{13}$$

当$V(x)=V_l(x)$时, 有

$$\begin{aligned}\dot{V}(x)=\dot{V}_l(x)=(\frac{\partial V_l(x)}{\partial x})^{\mathrm T}\dot{x}=\\ \sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j((\frac{\partial V_l(x)}{\partial x})^{\mathrm T}(A_i(x)-\\ B_i(x)F_{jl}(x))\hat{x}(x)).\end{aligned} \tag{14}$$

由于选择了最大型分段李雅普诺夫函数, 所以当$V(x)=V_l(x)$时有

$$\sum_{s=1}^{k}\lambda_{ils}(x)(V_l(x)-V_s(x))\geqslant 0, \tag{15}$$

其中$\lambda_{ils}(x)$为SOS多项式, 由式(11)保证. 且$V_l(x)>0$所以(由引理1)有

$$\begin{aligned}\dot{V}_l(x)\leqslant\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j((\frac{\partial V_l(x)}{\partial x})^{\mathrm T}(A_i(x)-\\ B_i(x)F_{jl}(x))\hat{x}(x)+\\ \sum_{s=1}^{k}\lambda_{ils}(x)(V_l(x)-V_s(x)))\leqslant 0.\end{aligned} \tag{16}$$

在李雅普诺夫意义下的稳定条件中, 李雅普诺夫函数对时间的导数需要负定, 即$\dot{V}(x)<0$. 因此, 本文引入正定项$-\alpha V_l(x)$ (其中α为负标量, 采用“$-\alpha$”的形式是为了后续编程的便利), 通过引理1得

$$\begin{aligned}\dot{V}_l(x)\leqslant\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j((\frac{\partial V_l(x)}{\partial x})^{\mathrm T}(A_i(x)-\\ B_i(x)F_{jl}(x))\hat{x}(x)-\alpha V_l(x)+\\ \sum_{s=1}^{k}\lambda_{ils}(x)(V_l(x)-V_s(x)))<0.\end{aligned} \tag{17}$$

所以当式(9)–(11)成立时, 有$V(x)<0$, 即系统稳定.

证毕.

注 1 当$V(x)=V_l(x)$时, 控制器切换至$u(t)=\sum_{i=1}^{r}h_i(z)F_{il}\,x(x)$, 并且由定理1所给条件保证了$V_l(x)<\alpha V_l(x)<0$. 当系统状态不在分段李雅普诺夫函数的分界点上时, 由定理1可直接得到$V_l(x)<0$, 即系统稳定. 当系统状态处于函数的切换边界上时, 也可以得到$V(x)\leqslant V_l(x)<0$, 证明如下:

证 假设t_0时刻$V_1(t_0)=V_2(t_0)$, 此时$l=\max\{1,2\}=2$, 且由定理1所给条件我们有$u(t)=\sum_{i=1}^{r}h_i(z)F_{i2}\hat{x}(x)$保证$V(x)=V_2(x)<0$, 所以当$V_2(t_0^-)>V_1(t_0^-)$时, 有

$$\dot{V}(t_0)=\lim_{t_0-t_0^-\to 0}\frac{V(t_0^-)-V(t_0)}{t_0^--t_0}=\dot{V}_2(t_0)<0,$$

当$V_2(t_0^-)<V_1(t_0^-)$即$V(t_0^-)=V_1(t_0^-)$时, 有

$$\dot{V}(t_0)=\lim_{t_0-t_0^-\to 0}\frac{V(t_0^-)-V(t_0)}{t_0^--t_0}=\dot{V}_1(t_0)=$$

$$\lim_{t_0-t_0^-\to 0}\frac{V_2(t_0^-)-V_1(t_0)}{t_0^--t_0}<\lim_{t_0-t_0^-\to 0}\frac{V_2(t_0^-)-V_2(t_0)}{t_0^--t_0}<0.$$

综上, 在分段多项式李雅普诺夫函数的函数分界点上依旧满足其导数小于零, 即系统依旧是稳定的. 证毕.

定理1所给条件中, 存在如$(\frac{\partial V_l(x)}{\partial x})^{\mathrm T}B_i(x)F_{jl}(x)$的非凸项, 所以该条件无法直接用现有的SOS 工具求解. 为了进行求解, 考虑如下非凸条件:

$$G(x)H(x)<0, \tag{18}$$

其中$G(x)$, $H(x)$为多项式矩阵, 且两者都是决策变量(矩阵). 为求解该式, 引入正定多项式矩阵$\psi(x)$将式(18)转换为

$$-G(x)H(x)+\alpha\psi(x)\geqslant 0, \tag{19}$$

其中α为标量, 且若在$\alpha<0$的情况下找到式(19)的解, 则式(18)也可求解. 同时式(19)可转变为SOS条件

$$-v^{\mathrm T}\{G(x)H(x)-\alpha\psi(x)\}v\text{是SOS}, \tag{20}$$

其中v是独立于x的向量. 又因为式(20)中存在$G(x)H(x)$的项, 所以条件(20)为非凸(双线性)条件, 无法用现有的SOS求解工具直接求解, 所以对$G(x)$, $H(x)$, $\psi(x)$假设非常微小的扰动$\delta G(x)$, $\delta H(x)$,

$\delta\psi(x)$. 即式(20)变为

$$-v^{\mathrm{T}}\{(G(x)+\delta G(x))(H(x)+\delta H(x))-\alpha(\psi(x)+\delta\psi(x))\}v\text{是SOS}, \tag{21}$$

又$\delta G(x)\delta H(x)$为微扰乘积, 相比于其他项很小, 故将其忽略不会对结果产生很大影响. 最终得到如下条件: $-v^{\mathrm{T}}\{G(x)H(x)+\delta G(x)H(x)+\delta H(x)G(x)-\alpha(\psi(x)+\delta\psi(x))\}v$是SOS.

将式(18)的求解转换为了基于SOS方法的最小凸优化问题:

$$\min_{\delta G(x),\delta H(x),\delta\psi} \alpha \ \text{ s.t. 式(22)–(26)},$$

$$-v_1^{\mathrm{T}}\{\psi(x)+\delta\psi(x)-\epsilon_1(x)\}v_1\text{是SOS}, \tag{22}$$

$$-v_2^{\mathrm{T}}\{G(x)H(x)+\delta G(x)H(x)+\delta H(x)G(x)-\alpha(\psi(x)+\delta\psi(x))\}v_2\text{是SOS}, \tag{23}$$

$$-v_3^{\mathrm{T}}\begin{bmatrix}\epsilon_g G^{\mathrm{T}}(x)G(x) & \delta G(x)\\ \delta G(x) & I\end{bmatrix}v_3\text{是SOS}, \tag{24}$$

$$-v_4^{\mathrm{T}}\begin{bmatrix}\epsilon_h H^{\mathrm{T}}(x)H(x) & \delta H(x)\\ \delta H(x) & I\end{bmatrix}v_4\text{是SOS}, \tag{25}$$

$$-v_5^{\mathrm{T}}\begin{bmatrix}\epsilon_\psi \psi^{\mathrm{T}}(x)\psi(x) & \delta\psi(x)\\ \delta\psi(x) & I\end{bmatrix}v_5\text{是SOS}, \tag{26}$$

其中: $\epsilon_g, \epsilon_h, \epsilon_{\mathrm{fi}}$都是很小的正数, $\epsilon_1(x)$为径向无界的正定多项式. 运用Schur补定理, 得到式(24)的等价条件$\epsilon_g G^{\mathrm{T}}(x)G(x)-\delta G(x)\delta G(x)^{\mathrm{T}}\geqslant 0$, 同理可看出, 当式(24)–(26)成立时, 保证了对小微扰$\delta G(x)$, $\delta H(x)$, $\delta\psi(x)$的假设. 若该最小优化问题在$a<0$时有解, 则求得了式(18)的解. 根据上述方法, 本文给出求解所用路径跟踪算法的具体步骤如下:

路径跟踪算法:

步骤 1　设$n=0$, 选取适当的N个正多项式, 作为初始李雅普诺夫函数$V_l^{(0)}(x)$.

步骤 2　设$V_l(x)=V_l^{(0)}(x)$, 并对如下优化问题进行求解:

$$\min_{F_j(x),\lambda_{ils}(x)} \alpha \ \text{ s.t. 式(9)–(11)}, \tag{27}$$

若求得的$\alpha<0$, 则所得解为定理1的可行解. 若该优化问题所得最优解$\alpha\geqslant 0$, 则将所得$F_{jl}(x)$, $\lambda_{ils}(x)$记为$F_{jl}^{(n)}(x)$, $\lambda_{ils}^{(n)}(x)$, 进行下一步.

步骤 3　令$F_{jl}(x)=F_{jl}^{(n)}(x)$, $\lambda_{ils}(x)=\lambda_{ils}^{(n)}(x)$求解下列最优化问题:

$$\min_{V_l(x)+\delta V_l(x)} \alpha \ \text{ s.t. 式(28)–(31)},$$

$$V_l(x)+\delta V_l(x)-\epsilon(x)\text{ 是 SOS},\ l\in\{1,\cdots,N\}, \tag{28}$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j((\frac{\partial V_l(x)+\delta V_l(x)}{\partial x})^{\mathrm{T}}(A_i(x)-B_i(x)F_{jl}(x))\hat{x}(x)+\alpha(V_l(x)+\delta V_l(x))-\sum_{s=1}^{k}\lambda_{ils}(x)((V_l(x)+\delta V_l(x))-(V_s(x)-\delta V_s(x))))\ \text{是SOS},\ l\in\{1,\cdots,N\}, \tag{29}$$

$$\lambda_{ils}(x)\text{是SOS},\ l,s\in\{1,\cdots,N\},\ i\in\{1,\cdots,r\}, \tag{30}$$

$$v^{\mathrm{T}}\begin{bmatrix}\epsilon_V V_l^2(x) & \delta V_l(x)\\ \delta V_l(x) & I\end{bmatrix}v\text{是SOS},\ l\in\{1,\cdots,N\}, \tag{31}$$

其中: ϵ_V为小实数, v为独立于x且维度适当的列向量. 进行下一步骤.

步骤 4　在步骤3中求解得$\delta V_l(x)$, 令$V_l^{(n+1)}(x)=V_l^{(n)}(x)+\delta V_l(x)$, $F_{jl}(x)=F_{jl}^{(n)}(x)$, $\lambda_{ils}(x)=\lambda_{ils}^{(n)}(x)$, 然后再令$n=n+1$, $V_l(x)=V_l^{(n)}(x)$并继续求解如下最优化问题:

$$\min_{\delta F_l(x)\ ,\ \delta\lambda_{ils}(x)} \alpha \ \text{ s.t. 式(32)–(36)}$$

$$V_l(x)-\epsilon(x)\text{是SOS},\ l\in\{1,\cdots,N\}, \tag{32}$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j((\frac{\partial V_l(x)}{\partial x})^{\mathrm{T}}(A_i(x)-B_i(x)(F_{jl}(x)+\delta F_{jl}(x)))\hat{x}(x)+\alpha V_l(x)-\sum_{s=1}^{k}(\lambda_{ils}(x)+\delta\lambda_{ils}(x))(V_l(x)-V_s(x)))\text{是SOS},\ l\in\{1,\cdots,N\}, \tag{33}$$

$$\lambda_{ils}(x)+\delta\lambda_{ils}(x)\text{是SOS},\ l,s\in\{1,\cdots,N\},\ i\in\{1,\cdots,r\}, \tag{34}$$

$$v^{\mathrm{T}}\begin{bmatrix}\epsilon_F F_{il}(x)F_{il}(x)^{\mathrm{T}} & \delta F_{il}(x)\\ \delta F_{il}(x)^{\mathrm{T}} & I\end{bmatrix}v\text{是SOS},\ l\in\{1,\cdots,N\}, i\in\{1,\cdots,r\}, \tag{35}$$

$$v^{\mathrm{T}}\begin{bmatrix}\epsilon_\lambda \lambda_{ils}^2(x) & \delta\lambda_{ils}(x)\\ \delta\lambda_{ils}(x) & 1\end{bmatrix}v\text{是SOS},\ l,s\in\{1,\cdots,N\}, i\in\{1,\cdots,r\}, \tag{36}$$

其中: ϵ_F, ϵ_λ为小实数, v为线性无关且独立于x的列向量. 若求得的$\alpha<0$, 则所得解为定理1的可行解, 记

$$F_{jl}(x)=F_{jl}^{(n-1)}(x)+\delta F_{jl}(x),$$
$$\delta\lambda_{ils}(x)=\lambda_{ils}^{(n-1)}(x)+\delta\lambda_{ils}(x)$$

跳出迭代. 若该优化问题所得最优解$\alpha\geqslant 0$, 则记

$$F_{jl}^{(n)}(x)=F_{jl}^{(n-1)}(x)+\delta F_{jl}(x),$$
$$\lambda_{ils}^{(n)}(x)=\lambda_{ils}^{(n-1)}(x)+\delta\lambda_{ils}(x),$$

然后进行步骤3.

注 2　假设$V_l(x)$的标量决策变量数为φ_V(例如$V_l(x)$

$= d_{1l}x_1^2 + d_{2l}x_1x_2 + d_{3l}x_2^2$, 则$\varphi_V = 3$, 其中$d_{1l}, d_{2l}, d_{3l}$为标量决策变量)$\lambda_{ils}$的标量决策变量数为$\varphi_\lambda$. $F_{jl}(x)$的标量决策变量数为φ_F. 算法的步骤2以及步骤4中分别含有$r\varphi_F + rN^2\varphi_\lambda$个标量决策变量数, 步骤3中则含有$N\varphi_V$个标量决策变量数. 另外步骤2中SOS条件数为$2N + rN^2$个, 步骤3中SOS条件数为$3N + rN^2$, 步骤4中SOS条件数为$r + 2N + 2rN^2$.

4 算例仿真

例 1 考虑如下二规则多项式模糊模型1[9].

模型规则i:

若x_i为M_l, 则

$$\dot{x} = A_i(x)\hat{x}(x) + B_i(x)u,\ i = 1, 2,$$

其中$\hat{x}(x) = x^{\mathrm{T}} = [x_1\ x_2]$, 系统矩阵如下:

$$A_1 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -\alpha & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172\alpha & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} x_1 \\ b \end{bmatrix},\ B_2 = \begin{bmatrix} x_1 \\ b \end{bmatrix},$$

当$a = 6$, $b = 20$时, 隶属函数给出如下:

$$h_1 = \frac{\sin x_1 + 0.2172x_1}{1.2172x_1},\ h_2 = \frac{x_1 - \sin x_1}{1.2172x_1},$$

应用本文定理1给出该模型的稳定性条件, 并通过设计的路径跟踪算法求解得到最终的迭代结果如下:

$$\begin{aligned}
V_1 &= 1319.64x_1^6 + 281.81x_1^5x_2 + 121.10x_1^4x_2^2 - \\
&\quad 6.95x_1^3x_2^3 + 24.30x_1^2x_2^4 + 0.89x_2^6, \\
V_2 &= 1266.83x_1^6 + 456.35x_1^5x_2 + 95.41x_1^4x_2^2 + \\
&\quad 5.75x_1^3x_2^3 + 41.55x_1^2x_2^4 + 0.95x_2^6, \\
V_3 &= 1266.13x_1^6 + 556.88x_1^5x_2 + 74.69x_1^4x_2^2 + \\
&\quad 0.20x_1^3x_2^3 + 38.64x_1^2x_2^4 + 0.93x_2^6, \\
F_{11} &= [1.4231x_1 + 0.12875x_2 - 0.66197 \\
&\quad 0.12875x_1 + 1.0747], \\
F_{21} &= [1.3826x_1 + 0.13805x_2 - 0.4667 \\
&\quad 0.13805x_1 + 1.0527], \\
F_{12} &= [1.4766x_1 + 0.010235x_2 - 0.76232 \\
&\quad 0.010235x_1 - 0.047205x_2 + 0.60819], \\
F_{22} &= [1.4222x_1 + 0.029532x_2 - 0.47067 \\
&\quad 0.029532x_1 - 0.049883x_2 + 0.57382], \\
F_{13} &= [1.3481x_1 + 0.088509x_2 - 0.56452 \\
&\quad 0.088509x_1 - 0.024991x_2 + 0.42607], \\
F_{23} &= [1.3349x_1 + 0.084061x_2 - 0.23954 \\
&\quad 0.084061x_1 - 0.02906x_2 + 0.43086].
\end{aligned}$$

从图1可以看出模型系统的全局渐近稳定性, 对随机选取的6个初始状态, 本文条件均能控制其归于零点. 6例初始状态分别选为Case1 $= [-7\ \ 8]^{\mathrm{T}}$, Case2 $= [-7\ \ 4]^{\mathrm{T}}$, Case3 $= [-6\ \ -4]^{\mathrm{T}}$, Case4 $= [8\ \ -8]^{\mathrm{T}}$, Case5 $= [8\ \ 0]^{\mathrm{T}}$, Case6 $= [7\ \ 8]^{\mathrm{T}}$. 整个状态空间(红色箭头)也显示出了汇聚(稳定)的趋势. 图2显示的是初始值为$[3\ \ 4]^{\mathrm{T}}$时分段多项式李雅普诺夫函数图像, 可以看出当$t_0 = 0.009$时分段多项式李雅普诺夫函数发生了切换, 此时设计的切换控制器也随之进行切换. 从图像可以看出分段李雅普诺夫函数满足本文设计稳定的要求, 且能较快稳定(趋于零).

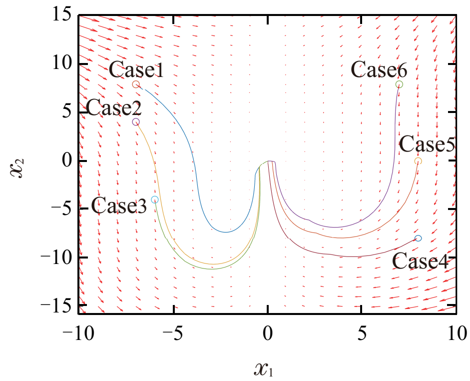

图 1 模型1在不同初始状态下的响应

Fig. 1 Behaviors of model 1 under different initial states

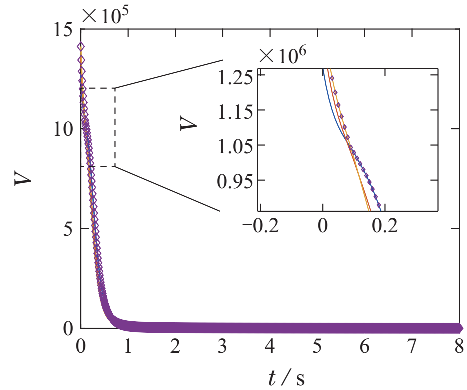

图 2 模型1的分段李雅普诺夫函数

Fig. 2 PPLF of model 1

例 2 考虑如下三规则多项式模糊模型2[27].

模型规则i:

若x_i是M_l, 则

$$\dot{x} = A_i(x)\hat{x}(x) + B_i(x)u,\ i = 1, 2, 3,$$

其中$\hat{x}(x) = x^{\mathrm{T}} = [x_1\ \ x_2]$; 系统矩阵如下:

$$A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix},$$

$$A_3=\begin{bmatrix}-a & -4.33\\ 0 & 0.05\end{bmatrix},$$

$$B_1=\begin{bmatrix}1\\ b\end{bmatrix},\ B_2=\begin{bmatrix}8\\ b\end{bmatrix},\ B_3=\begin{bmatrix}-b+6\\ -1\end{bmatrix},$$

隶属函数给出如下:

$$h_1(x)=\frac{\cos(10x_1)+1}{4},\ h_2(x)=\frac{\sin(10x_1)+1}{4},$$

$$h_3(x)=\frac{\cos(10x_1)-\sin(10x_1)+2}{4},$$

当$a=2$, $b=7.5$, 并且$N=2$时, 通过本文的路径跟踪算法, 得到如下可行解:

$$\begin{aligned}V_1=\ &0.000191x_1^8-0.000787x_1^7x_2+0.00958x_1^6x_2^2-\\ &0.01236x_1^5x_2^3+0.1224x_1^4x_2^4+0.5058x_1^3x_2^5+\\ &2.8735x_1^6x_2^2+4.4848x_1x_2^7+13.9x_2^8,\\ V_2=\ &0.000268x_1^8-0.000891x_1^7x_2+0.0115x_1^6x_2^2-\\ &0.01338x_1^5x_2^3+0.11766x_1^4x_2^4+0.52774x_1^3x_2^5+\\ &3.1022x_1^6x_2^2+4.8389x_1x_2^7+14.8954x_2^8,\\ F_{11}=\ &[5.558\ \ -1.982],\ F_{12}=[5.801\ \ -1.706],\\ F_{21}=\ &[17.884\ \ 20.688],\ F_{22}=[16.233\ \ 18.876],\\ F_{31}=\ &[-4.557\ \ -66.206],\\ F_{32}=\ &[-4.146\ \ -60.349].\end{aligned}$$

从图3给出的是在状态空间中6个随机初始状态的响应轨迹. 可以看出6个随机初始状态在本文所设计的控制器控制下都是渐近稳定的, 可以判断定理1方法的有效性. 6例初始状态分别为Case1 $=[-7\ \ 8]^{\mathrm{T}}$, Case2 $=[-6\ \ 4]^{\mathrm{T}}$, Case3 $=[-6\ \ -4]^{\mathrm{T}}$, Case4 $=[8\ \ -8]^{\mathrm{T}}$, Case5 $=[9\ \ 0]^{\mathrm{T}}$, Case6 $=[7\ \ 8]^{\mathrm{T}}$. 图4列出了$a=2$时本文与文献[27, 30, 37–39]中方法所得宽松度的比较, b值从0到8间隔0.5展示各个方法的最大b值. 从图示结果可以看出, 相较于同样采用了分段李雅普诺夫函数的文献[27], 本文所给稳定条件的解更为宽松. 文献[30]的研究对象同样是多项式模糊模型, 可以看出本文定理1取得了与文献[30]在$s=2$时的相等的宽松度, 而根据文献[30]中所述, s的增大会带来更重的计算负担, 而当$s=0$时其$b_{\max}=7$, 所以与文献[30]相比, 本文所给方法能在不引入额外计算负担的情况下提高解的宽松度.

5　总结

本文就闭环多项式模糊系统稳定性进行研究, 选用了最大型分段多项式李雅普诺夫函数, 给出了基于SOS方法的稳定性条件, 本文采用路径跟踪算法对非凸问题进行迭代求解. 另外, 对于最大型分段多项式李雅普诺夫函数在函数切换点上的稳定性, 本文也作出了相关证明. 最后, 两个算例的仿真求解也从实践层面上验证了本文所给条件的可行性与优越性. 后续工作, 将围绕改进控制器的设计, 考虑隶属函数信息等展开, 进一步提高稳定条件的宽松度.

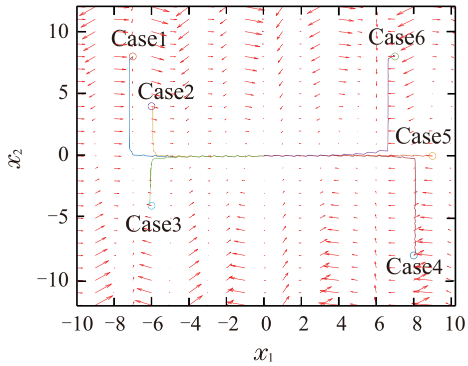

图 3　模型2在不同初始状态下的响应

Fig. 3　Behaviors of model 2 under different initial states

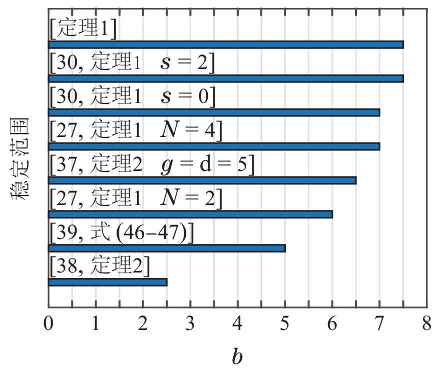

图 4　不同稳定方法对应最大的b值

Fig. 4　Maximun feasible b for different stabilization criteria

参考文献:

[1] TAKAGI T, SUGENO M. Fuzzy identification of systems and its applications to modeling and control, *IEEE Transactions on Systems, Man, and Cybernetics*, 1985, 15(1): 116 – 132.

[2] GUERRA T M, SALA A, TANAKA K. Fuzzy control turns 50: 10 years later. *Fuzzy Sets and Systems*, 2015, 281: 168 – 182.

[3] LAM H K, LEUNG F H F, Stability analysis of discrete-time fuzzy-model-based control systems with time delay: Time delay-independent approach. *Fuzzy Sets and Systems Theme: Fuzzy and Non-linear Control*, 2008, 159(8): 990 – 1000.

[4] TANAKA K, IKEDA T,WANG H O. Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, 1998, 6(2): 250 – 265.

[5] SALA A, ARIÑO C. Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, 2007, 158(24): 2671 – 2686.

[6] LIU Y S, FANG C H. A new LMI-based approach to relaxed quadratic stabilization of T-S fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 2006, 14(3): 386 – 397.

[7] EUNTAI K, HEEJIN L. New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 2000, 8(5): 523 – 534.

[8] TANAKA K, OHTAKE H, WANG H O. Guaranteed cost control of polynomial fuzzy systems via a sum of squares approach. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 2009, 39(2): 561 – 567.

[9] TANAKA K, YOSHIDA H, OHTAKE H, et al. A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 2009, 17(4): 911 – 922.

[10] TANAKA K, YOSHIDA H, OHTAKE H, et al. A sum of squares approach to stability analysis of polynomial fuzzy systems. *American Control Conference*. New York, USA: IEEE, 2007: 4071 – 4076.

[11] TANAKA K, YOSHIDA H, OHTAKE H, et al. Stabilization of polynomial fuzzy systems via a sum of squares approach. *The 22nd International Symposium on Intelligent Control*. Singapore: IEEE, 2007: 160 – 165.

[12] SALA A, ARIO C. Polynomial fuzzy models for nonlinear control: A taylor series approach. *IEEE Transactions on Fuzzy Systems*, 2009, 17(6): 1284 – 1295.

[13] PAPACHRISTODOULOU A, ANDERSON J, VALMORBIDA G, et al. SOSTOOLS version 4.00 sum of squares optimization toolbox for MATLAB. *Arxiv Preprint*, 2013: arxiv:1310.4716.

[14] SEILER P, SOSOPT. A toolbox for polynomial optimization. *Arxiv Preprint*, 2013: arXiv:1308.1889v1.

[15] LAM H K, NARIMANI M, LI H, et al. Stability analysis of polynomial-fuzzy-model-based control systems using switching polynomial lyapunov function. *IEEE Transactions on Fuzzy Systems*, 2013, 21(5): 800 – 813.

[16] NARIMANI M, LAM H K. SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions. *IEEE Transactions on Fuzzy Systems*, 2010, 18(5): 862 – 871.

[17] LAM H K, NARIMANI M. Sum-of-squares-based stability analysis of polynomial fuzzy-model-based control systems. *IEEE International Conference on Fuzzy Systems*. Jeju, Korea (South): IEEE, 2009: 234 – 239.

[18] LI L Z, TANAKA K. Relaxed sum-of-squares approach to stabilization of polynomial fuzzy systems. *International Journal of Control, Automation and Systems*, 2021, 19(8): 2921 – 2930.

[19] ZHANG H, WANG Y, WANG Y, et al. A novel sliding mode control for a class of stochastic polynomial fuzzy systems based on SOS method. *IEEE Transactions on Cybernetics*, 2020, 50(3): 1037 – 1046.

[20] ZHAO Y, HE Y, FENG Z, et al. Relaxed sum-of-squares based stabilization conditions for polynomial fuzzy-model-based control systems. *IEEE Transactions on Fuzzy Systems*, 2019, 27(9): 1767 – 1778.

[21] YE D, LI X. State feedback controller design for general polynomial-fuzzy-model-based systems with time-varying delay. *International Journal of Systems*, 2019, 21(2): 583 – 591.

[22] SOL K H, BAE P J, HOON J Y. A fuzzy lyapunov–krasovskii functional approach to sampled-data output-feedback stabilization of polynomial fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 2018, 26(1): 366 – 373.

[23] GASSARA H, HAJJAJI AE, KRID M, et al. Stability analysis and memory control design of polynomial fuzzy systems with time delay via polynomial lyapunov-krasovskii functional. *International Journal of Control, Automation and Systems*, 2018, 16(4): 2011 – 2020.

[24] FURQON R, CHEN Y J, TANAKA M, et al. An SOS-based control lyapunov function design for polynomial fuzzy control of nonlinear systems. *IEEE Transactions on Fuzzy Systems*, 2017, 25(4): 775 – 787.

[25] TANAKA K, TANAKA M, CHEN Y, et al. A new sum-of-squares design framework for robust control of polynomial fuzzy systems with uncertainties. *IEEE Transactions on Fuzzy Systems*, 2016, 24(1): 94 – 110.

[26] HASSIBI A, HOW J, BOYD S. A path-following method for solving BMI problems in control. *Proceedings of the American Control Conference*. San Diego, CA, USA: IEEE, 1999, 2: 1385 – 1389.

[27] CHEN Y, OHTAKE, TANAKA K, et al. Relaxed stabilization criterion for T–S fuzzy systems by minimum-type piecewise-lyapunov-function-based switching fuzzy controller. *IEEE Transactions on Fuzzy Systems*, 2012, 20(6): 1166 – 1173.

[28] CHEN Y, TANAKA M, TANAKA K, et al. Stability analysis and region-of-attraction estimation using piecewise polynomial Lyapunov functions: Polynomial fuzzy model approach. *IEEE Transactions on Fuzzy Systems*, 2015, 23(4): 1314 – 1322.

[29] HU T, MA L, LIN Z L. Stabilization of switched systems via composite quadratic functions. *IEEE Transactions on Automatic Control*, 2008, 53(11): 2571 – 2585.

[30] CHEN Y, TANAKA K, TANAKA M, et al. A novel path-following-method-based polynomial fuzzy control design, *IEEE Transactions on Cybernetics*, 2021, 51(6): 2993 – 3003.

[31] TANAKA K, OHTAKE H, YAMAGUCHI T, et al. Stability analysis of nonlinear systems via multiple mixed max-min based lyapunov functions. *International Conference on Fuzzy Systems*. Barcelona, Spain: IEEE, 2010: DOI: 10.1109/FUZZY.2010.5584607.

[32] TAN W, PACKARD A. Stability region analysis using sum of squares programming. *American Control Conference*. Minneapolis, MN, USA: IEEE, 2006: 1 – 6.

[33] CHEN Y, TANAKA M, TANAKA K, et al. Piecewise polynomial Lyapunov functions based stability analysis for polynomial fuzzy systems. *International Conference on Control System, Computing and Engineering*. Penang, Malaysia: IEEE, 2013: 34 – 39.

[34] BOYD S, EL GHAOUI L, FERON E, et al. *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: Society for Industrial and Applied Mathematics, 1994.

[35] TANAKA K, WANG H O. Fuzzy control systems design and analysis: A linear matrix inequality approach. *Automatica*, 2003, 39(11): 2011 – 2013.

[36] OHTAKE H, TANAKA K, WANG H O. Switching fuzzy controller design based on switching Lyapunov function for a class of nonlinear systems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 2006, 36(1): 13 – 23.

[37] MONTAGNER V F, OLIVEIRA R C L F, PERES P L D. Convergent LMI relaxations for quadratic stabilizability and H_∞ control of takagi-sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 2009, 17(4): 863 – 873.

[38] LIU X, ZHANG Q. Approaches to quadratic stability conditions and H_∞ control designs for T-S fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 2003, 11(6): 830 – 839.

[39] TEIXEIRA M C M, ASSUNCAO E, AVELLAR R G. On relaxed LMI-based designs for fuzzy regulators and fuzzy observers. *IEEE Transactions on Fuzzy Systems*, 2003, 11(5): 613 – 623.

作者简介:

吴雨棠 硕士研究生, 目前研究方向为基于多项式模糊模型的系统稳定控制, E-mail: wyt980123@163.com;

李丽珍 副教授, 硕士生导师, 目前研究方向为多维系统的模糊系统、多维网络化控制系统、最优控制, E-mail: lilizhen@shiep.edu.cn;

胡德时 硕士研究生, 目前研究方向基于平方和的多项式模糊控制系统, E-mail: 2097552846@qq.com.