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Riemann-Liouville型分数阶导数的非线性估计

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摘要: 本文主要研究任意有界连续信号的Riemann-Liouville分数阶导数估计问题. 当分数阶α属于0到1 时, 首先利用滑模技术提出一种有界连续信号分数阶导数的非线性估计方法; 然后将其结果推广至分数阶$\alpha \in \mathbb{R}^{+}$的情况, 并给出相应的非线性估计方案. 借助Riemann-Liouville分数阶微积分频率分布模型, 本文详细分析讨论了所给分数阶导数非线性估计的收敛性问题, 并得到相应闭环系统是渐近稳定的结论. 文中所提方法的主要优点是在事先未知给定信号分数阶导数上界的情况下, 不仅能自适应地估计其Riemann-Liouville分数阶导数, 而且当信号中含有随机噪声和不确定扰动时依然能正常工作. 数值仿真实例验证了本文所给估计方法的可行性和有效性.

关键词: 分数阶微积分; Riemann-Liouville; 非线性系统; 自适应滑模; Gaussian白噪声

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Nonlinear estimation of Riemann-Liouville type fractional-order derivative

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Abstract: This paper mainly concerns about the problem of estimation of the Riemann-Liouville fractional derivative of arbitrarily bounded continuous signal. By using sliding mode technique, a nonlinear fractional-order derivative estimator of a bounded continuous signal for the order α between 0 and 1 is proposed firstly. Then it is extended to the case of arbitrary order $\alpha \in \mathbb{R}^{+}$, and the corresponding estimation scheme is also established. The convergence of the presented estimator is discussed in more detail with the assistance of frequency distributed model of the Riemann-Liouville fractional calculus. Meanwhile the matching closed-loop plant is asymptotically stable. The major advantages of the proposed methodology can not only adaptively estimate the Riemann-Liouville fractional derivative of a given signal that is not clear about the upper bound of fractional derivative itself in advance, but also adapt to the uncertain disturbances or stochastic noise environment in system. Numerical simulation results of an example are used to verify the practicality and availability of our given estimation scheme.

Key words: fractional calculus; Riemann-Liouville; nonlinear system; adaptive sliding mode; white Gaussian noise

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1 Introduction

Fractional calculus and its applications are one of the most rapidly developing fields of science and engineering in the last two decades. Numerous excellent references such as books, review articles, and conference proceedings have been obtained in fractional-order (FO) community [1–9]. It is widely believed that in the integer-order (IO) systems the time derivatives of a signal are usually used in the system theories and engineering applications. For example, the velocity and acceleration of a target need to be estimated and measured by the time derivative of the given signal in the radar applications. This is also the case for the FO systems. The most wonderful example is the design procedures

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of fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller [10–13] and fractional sliding-mode controller [14–18], in which the fractional calculus of a error signal is desired to synthesize the input signal of the controlled systems. For some of the basic elementary function, however, how to calculate its FO derivative and give the analytic expression of its fractional derivative are continuing to experience difficulty since the complexity of fractional calculus. Most of the researchers [19–22] took numerical computation method to calculate and estimate fractional integral and derivative of a given signal. Whereas, it would occur some limitations which are derived from the existence of uncertain disturbances or stochastic noise. That is, the method is a result of the numerical theory, which may not be available to obtain the estimated values (or accurate values) of fractional derivatives of a contaminated signal.

In addition, we know that in the classical IO systems sliding mode control (SMC) technique can provide a robust state observer of systems [23–28], and even estimate multiple differentiations of the system signals given in real time. In the FO control plant, the information of the system states usually needs to be accessed to construct the appropriate controller. That is to say, construction of a special FO integrator and differentiator can not be avoided in the face of fractional calculus and FO systems. Up to now, however, only a few articles [29] investigated the issue of estimation of fractional calculus by using the control theoretical framework. Nevertheless, it is noted that since the weak singularity of definition of fractional calculus (i.e. the existence of kernel $(t-\tau)^{\alpha-1}$ or $(t-\tau)^{n-\alpha-1}$), the method of estimated value about IO derivative can not extend to the case of FO directly. And in the plant of control, it may have to be concerned about the problem of initial conditions and initialization of FO systems. Therefore, for estimation of fractional derivative of a signal with and without noise, this is still an issue which is worth to be further studied and explored.

Motivated by the aforementioned statement in the paper, we start by considering the problem of estimation of Riemann-Liouville (RL) fractional derivative with FO α between 0 and 1, then extend the corresponding results to the case of arbitrary order (i.e. $\alpha \in \mathbb{R}^+$). By using sliding mode technique, a fractional estimator can be applied to estimate the FO derivative of a bounded smooth nonlinear signal. The convergence of the proposed fractional estimator is assured by employing the stability theory of the continuous frequency distributed model of RL fractional calculus. To remove the condition of the upper bound for the fractional derivative of observed signals, further an adaptive SMC algorithm is also proposed. Meanwhile, the switching gain is adequately adjusted and controlled online by the constructed adaptive law. Finally, to illustrate the validity and availability of the presented scheme, the estimation of fractional derivative of a smooth signal with and without involving noise is exhibited in an example with numerical simulations and analyses.

The major contributions of this article are stated as follows: 1) From the control framework, the method about the estimation of RL's fractional derivative of a given signal is derived. 2) The FO derivative estimator is in a position to adapt to the uncertain disturbances or stochastic noise in system. 3) Compared with the existing research, the issue of initial value of diffusive representation for RL fractional calculus is illustrated.

The structure of this paper is constructed as follows. Some basic lemmas on fractional calculus are presented in Section 2. Based on the control theoretical framework, estimation of fractional derivative is found in Section 3. Simulation results and conclusion are given separately in Section 4 and Section 5.

2 Preliminaries

In this section, to discuss the problem of estimation of RL fractional derivative, some basic lemmas on fractional calculus are presented. For convenience, the lower limit of fractional calculus in the theoretical analysis section of this paper is assumed to be zero. Whereas, the lower limit is still chosen as t_0 in the numerical simulation.

Lemma 1 If any order $\alpha \in \mathbb{R}^+$ and $f(t) \in L_p(0,b)$, $(1 \leqslant p \leqslant \infty)$, then the following equalities

$$ {}_0^{\mathrm{RL}}D_t^{\alpha}[{}_0I_t^{\alpha} f(t)] = f(t), \tag{1}$$

and

$$ {}_0I_t^{\alpha}({}_0^{\mathrm{RL}}D_t^{\alpha} f(t)) = f(t) - \sum_{k=1}^{n} \frac{t^{\alpha-k}[{}_0^{\mathrm{RL}}D_t^{\alpha-k} f(t)]_{t=0}}{\Gamma(\alpha-k+1)} \tag{2}$$

hold. Clearly, if $0 \leqslant \alpha < 1$, then

$$ {}_0I_t^{\alpha}({}_0^{\mathrm{RL}}D_t^{\alpha} f(t)) = f(t) - \frac{t^{\alpha-1}[{}_0^{\mathrm{RL}}D_t^{\alpha-1} f(t)]_{t=0}}{\Gamma(\alpha)}, \tag{3}$$

where ${}_0I_t^{\alpha}(\cdot)$ and ${}_0^{\mathrm{RL}}D_t^{\alpha}(\cdot)$ are usually referred to as RL's α-th fractional integration and derivation, respectively. More information about fractional calculus of RL can be found in many references. In what follows, an important formula will be presented here, which will be used later in the simulation.

Lemma 2 If there is a RL fractional calculus ${}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1} f(t)$, $(0<\alpha<1)$, then the formula ${}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1} f(t) = {}_0^{\mathrm{RL}}D_{\epsilon}^{\alpha-1} f(\epsilon+t_0)$ holds, for any initial time t_0.

Proof From the concept of RL fractional calculus, we have

$$ {}_{t_0}^{\mathrm{RL}}D_t^{\alpha-1} f(t) = {}_{t_0}I_t^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau. $$

Setting $\tau = \varsigma + t_0$, then it yields $\mathrm{d}\tau = \mathrm{d}\varsigma$,

$\varsigma \in [0,\ t-t_0]$. Hence, one has

$$ {}^{\mathrm{RL}}_{t_0}D_t^{\alpha-1}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau= \frac{1}{\Gamma(1-\alpha)}\int_0^{t-t_0}\frac{f(\varsigma+t_0)}{(t-\varsigma-t_0)^{\alpha}}\mathrm{d}\varsigma. $$

Again setting $t=\epsilon+t_0$, then it follows that

$$ {}^{\mathrm{RL}}_{t_0}D_t^{\alpha-1}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^{t-t_0}\frac{f(\varsigma+t_0)}{(t-\varsigma-t_0)^{\alpha}}\mathrm{d}\varsigma= \frac{1}{\Gamma(1-\alpha)}\int_0^{\epsilon}\frac{f(\varsigma+t_0)}{(\epsilon-\varsigma)^{\alpha}}\mathrm{d}\varsigma= {}_0I_\epsilon^{1-\alpha}f(\epsilon+t_0)={}^{\mathrm{RL}}_{0}D_\epsilon^{\alpha-1}f(\epsilon+t_0). \qquad \square $$

Remark 1 The above Lemma can be extended to R-L's fractional derivation of arbitrary order $\alpha\in\mathbb{R}^+$, for any initial time t_0, i.e. ${}^{\mathrm{RL}}_{t_0}D_t^{\alpha}f(t)={}^{\mathrm{RL}}_{0}D_\epsilon^{\alpha}f(\epsilon+t_0)$, $(n-1<\alpha\leqslant n\in Z^+)$ because of $\mathrm{d}t=\mathrm{d}\epsilon$ and $\epsilon=t-t_0$ in the foregoing proof.

Lemma 3 Consider RL type FO differential equation with the initial condition

$$ \begin{cases} {}^{\mathrm{RL}}_{0}D_t^{\alpha}x(t)=f(t,\ x),\ \forall\alpha\in(0,\ 1),\\ [{}^{\mathrm{RL}}_{0}D_t^{\alpha-1}x(t)]_{t=0}=x_0, \end{cases} \tag{4} $$

where $f(t,\ x):[0,+\infty]\times D\to\mathbb{R}$ is a nonlinear function, and $D\subset\mathbb{R}$ is defined as the set of continuous functions $x:[0,+\infty]\to\mathbb{R}$. Then, Eq. (4) can be expressed as

$$ \begin{cases} \dfrac{\partial\varrho(t,\omega)}{\partial t}=-\omega\varrho(t,\omega)+f(t,x),\\ x(t)=\dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^{+\infty}\omega^{-\alpha}\varrho(t,\omega)\mathrm{d}\omega, \end{cases} \tag{5} $$

with $\varrho(0,\omega)=x_0$, where $\varrho(t,\omega)$ is called the frequency distributed state variable.

Proof Taking RL's FO integral of both sides of the first relation of Eq. (4), it follows from the formula (3) of Lemma 1 that

$$ x(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x_0={}_0I_t^{\alpha}f(t,x). $$

From the definition of RL's integration, one has

$$ {}_0I_t^{\alpha}f(t,x)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(\tau,x)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{+\infty}\mathrm{e}^{-s}s^{-\alpha}\mathrm{d}s\int_0^t\frac{f(\tau,x)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau= \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\int_0^t f(\tau,x)(\frac{t-\tau}{s})^{\alpha}\frac{\mathrm{e}^{-s}}{t-\tau}\mathrm{d}\tau\mathrm{d}s. $$

Employing the substitution $s=\omega(t-\tau)$, then $\mathrm{d}s=(t-\tau)\mathrm{d}\omega$. Consequently, we have

$$ {}_0I_t^{\alpha}f(t,x)= \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\omega^{-\alpha}\int_0^t\mathrm{e}^{-\omega(t-\tau)}f(\tau,x)\mathrm{d}\tau\mathrm{d}\omega. $$

That is,

$$ x(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x_0+\int_0^{+\infty}\kappa(\omega)\int_0^t\mathrm{e}^{-\omega(t-\tau)}f(\tau,\ x)\mathrm{d}\tau\mathrm{d}\omega, $$

where $\kappa(\omega)=\dfrac{\sin(\alpha\pi)}{\pi}\omega^{-\alpha}$.

And, $\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ satisfies the following equality:

$$ \frac{t^{\alpha-1}}{\Gamma(\alpha)}=\frac{t^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{+\infty}s^{-\alpha}\mathrm{e}^{-s}\mathrm{d}s= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{+\infty}(\frac{s}{t})^{-\alpha}\frac{\mathrm{e}^{-s}}{t}\mathrm{d}s= \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\omega^{-\alpha}\mathrm{e}^{-\omega t}\mathrm{d}\omega=\int_0^{+\infty}\kappa(\omega)\mathrm{e}^{-\omega t}\mathrm{d}\omega. \tag{6} $$

Thus, we have

$$ x(t)=\int_0^{+\infty}\kappa(\omega)(\mathrm{e}^{-\omega t}x_0+\int_0^t\mathrm{e}^{-\omega(t-\tau)}f(\tau,x)\mathrm{d}\tau)\mathrm{d}\omega. $$

Setting $\varrho(0,\ \omega)=x_0$, it can be verified that $\varrho(t,\ \omega)=\mathrm{e}^{-\omega t}\varrho(0,\omega)+\int_0^t\mathrm{e}^{-\omega(t-\tau)}f(\tau,x)\mathrm{d}\tau$ is the solution of the differential equation

$$ \frac{\partial\varrho(t,\omega)}{\partial t}=-\omega\varrho(t,\omega)+f(t,x), \tag{7} $$

with the initial condition $\varrho(0,\omega)$. Subsequently, it follows that

$$ x(t)=\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\omega^{-\alpha}\varrho(t,\omega)\mathrm{d}\omega. \tag{8} $$

Hence, the combination (7) and (8) results in Eq. (4). $\square$

Remark 2 In this Lemma, x_0 may be not equal to $x(0)$. This is so because

$$ x(0)=\int_0^{+\infty}\kappa(\omega)\varrho(0,\omega)\mathrm{d}\omega=\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\omega^{-\alpha}x_0\mathrm{d}\omega= \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\omega^{-\alpha}\left[{}^{\mathrm{RL}}_{0}D_t^{\alpha-1}x(t)\right]_{t=0}\mathrm{d}\omega. $$

And, $\int_0^{+\infty}\omega^{-\alpha}\mathrm{d}\omega=\int_0^1\omega^{-\alpha}\mathrm{d}\omega+\int_1^{+\infty}\omega^{-\alpha}\mathrm{d}\omega$. The first integral equals $\dfrac{1}{1-\alpha}$ as $0<\alpha<1$, while the second integral equals $\dfrac{1}{\alpha-1}$ as $\alpha>1$. Thus the integration approaches infinitely great, for any value ω. That is, there is no consistent link between $x(0)$ and x_0 except for $x(0)=0$.

Remark 3 The proof avoids using a Bromwich contour [30] to show the frequency distributed plant of the RL's fractional derivation. Furthermore, the relationship between the initial condition of RL's FO differential equation and the initial state of frequency distributed plant is illustrated explicitly in the lemma. Numerical algorithm of corresponding approximation of the formula (5), because the equivalent representation of Lemma 3 cannot be directly used, are found in the references [30–32]. But we will only point out one main difference; that is, their initial values satisfy the relation $x_0=[\bar\varrho(0,\omega_0)\ \ \cdots\ \ \bar\varrho(0,\omega_m)]$, where $\bar\varrho(0,\omega_i)=$

$[\bar{\varrho}_1(0,\omega_i)\ \ \cdots\ \ \bar{\varrho}_N(0,\omega_i)]^{\mathrm{T}}$, $(i=0,1,2,\ \cdots,\ m)$.

In what follows, we will present estimation of RL's fractional derivative with the order α by applying SMC technology.

3 Estimation of fractional derivative

3.1 The case of order $\alpha \in (0,1)$

The FO derivatives of a signal plays a central role in the theory and engineering applications of fractional domain. In view of the complexity of fractional calculus, we here start by considering the construction of fractional estimator of a smooth signal $r(t)$ and setting $|{}_0^{\mathrm{RL}}D_t^{\alpha}r(t)| \leqslant d_l$, for any $\alpha \in (l-1,\ l)$, $l=1,2,\cdots$, in which d_l is a known positive constant. And the following assumption is introduced.

Assumption 1 Suppose that RL's α-th fractional derivation of the signal $r(t)$ is bounded; that is, there exists a real constant d_1 such that $|{}_0^{\mathrm{RL}}D_t^{\alpha}r(t)| \leqslant d_1$, for any $\alpha \in (0,1)$, where d_1 is a known positive constant.

Apparently, if we will view $r(t)$ as the input signal of the estimator, then the output $x(t)$ could be supposed to be the estimation of ${}_0^{\mathrm{RL}}D_t^{\alpha}r(t)$. Since ${}_0^{\mathrm{RL}}D_t^{\alpha}r(t)$ is generally not easy to access and measure, we present the following model:

$$\begin{cases} e(t) = r(t) - {}_0I_t^{\alpha}x(t),\ \forall\alpha \in (0,1), \\ x(t) = u(t), \end{cases} \tag{9}$$

such that $e(t)$ can approach zero as time progresses. Subsequently, it concludes from the formula (1) Lemma 1 that $x(t)$ tends to ${}_0^{\mathrm{RL}}D_t^{\alpha}r(t)$, $(0<\alpha<1)$. By the linearity of the FO differentiation operator and Lemmas 1–3, for $\forall\alpha \in (0,\ 1)$, one gets

$$\begin{cases} \dfrac{\partial\phi(t,\omega)}{\partial t} = -\omega\phi(t,\omega) + {}_0^{\mathrm{RL}}D_t^{\alpha}r(t) - u(t), \\ e(t) = \displaystyle\int_0^{+\infty} \kappa(\omega)\phi(t,\omega)\mathrm{d}\omega, \end{cases} \tag{10}$$

with $\phi(0,\ \omega) = e_0$, where $\kappa(\omega) = \dfrac{\sin(\alpha\pi)}{\pi}\omega^{-\alpha}$. Now our aim is to devise a control law $u(t)$ such that the system (10) is asymptotically stable. Toward that end, a simple control law is described by the following sketch diagram[1], as shown in Fig. 1.

Its mathematical description is described as follows:

$$u(t) = k_0 g(e(t)) + k_1\mathrm{sgn}(e(t)),\ (k_1 \geqslant d_1), \tag{11}$$

where $g(\cdot)$ satisfies: 1) $g(e)=0$, for the point $e=0$; 2) $eg(e)>0$, for any $e \in (-\infty,0)\cup(0,+\infty)$. And, in the rest of the artical $g(\cdot)$ might be chosen as $\mathrm{e}^{q/p}$, ($p>q$ and p, q are odd), or a common sign function. For other objects, a possible $g(\cdot)$ may conform to the above conditions, but we will focus more attention on the design methodology of this paper.

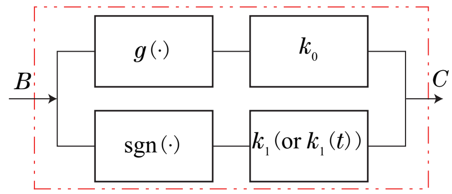


Fig. 1 A sketch diagram of controller u

To show the asymptotical stability of the controlled system (10), the following theorem is proposed.

Theorem 1 Under the control law (11), the closed-loop system (10) is asymptotically stable for any constant $k_1 \geqslant d_1$.

Proof Let us employ a Lyapunov function (or energy-like function) candidate $V(t,\omega) = \dfrac{1}{2}\displaystyle\int_0^{\infty}\kappa(\omega)\phi^2 \times (t,\omega)\mathrm{d}\omega$, for all $\omega \in (0,+\infty)$, which is obviously positive definite. Then the derivative of V is given by

$$\begin{aligned} \dot{V} = &-\int_0^{\infty}\omega\kappa(\omega)\phi^2(t,\omega)\mathrm{d}\omega + e(t)\,[{}_0^{\mathrm{RL}}D_t^{\alpha}r(t) - \\ &k_0 g(e(t)) - k_1\mathrm{sgn}(e(t))] \leqslant \\ &-\int_0^{\infty}\omega\kappa(\omega)\psi^2(t,\omega)\mathrm{d}\omega - k_0 e(t)g(e(t)) < \\ &-k_0 e(t)g(e(t)). \end{aligned} \tag{12}$$

Because

$$\omega\kappa(\omega) = \frac{\sin(\alpha\pi)}{\pi}\omega^{1-\alpha} > 0,\ \forall\alpha \in (0,1),$$

for all $\omega \in (0,+\infty)$. Thus, $\dot{V}(t,\ \omega)$ is negative definite. It Subsequently can be concluded that the system (10) is asymptotically stable. □

Remark 4 In the above proof, an important relation is used $[{}_0^{\mathrm{RL}}D_t^{\alpha-1}e(t)]_{t=0} = [{}_0^{\mathrm{RL}}D_t^{\alpha-1}r(t)]_{t=0}$.

In fact, following the first equation of (9), one gets

$$\begin{aligned} {}_0^{\mathrm{RL}}D_t^{\alpha-1}e(t) = {}&{}_0^{\mathrm{RL}}D_t^{\alpha-1}r(t) - {}_0^{\mathrm{RL}}D_t^{\alpha-1}[{}_0I_t^{\alpha}x(t)] = \\ &{}_0^{\mathrm{RL}}D_t^{\alpha-1}r(t) - \int_0^t x(\tau)\mathrm{d}\tau, \end{aligned}$$

where $\int_0^t x(\tau)\mathrm{d}\tau$ equals 0 at $t=0$. In addition, the lower limit of the frequency ω of Lemma 3 and Theorem 1 is usually exceedingly small, and reads 0 rarely. When $\omega=0$, however, the formula $e(t)=0$ holds for any $\int_0^{+\infty}\omega\kappa(\omega)\mathrm{d}\omega$ or $\int_0^{+\infty}\kappa(\omega)\mathrm{d}\omega$.

Remark 5 In order to get access to the more excellent performances of the estimation and further filter out a few vibrations and chatterings which exist possibly in the estimation, we can make use of the filter $F(s) = \dfrac{1}{\vartheta s^{\beta}+1}$, in which $\beta \in (0,1]$. Magnitude-frequency and phase-frequency characteristics of $F(s)$ from a variety of β and ϑ is shown in Fig. 2. Judging from the figure, we can see that the anti-interference

[1] With respect to the performance of the diagram in the entire design, more information can be illustrated by Appendix which is only an approximate analysis that follows closely the references [25,27].

performance of the filter becomes more competent as β increase. How to choose a good value for the parameter ϑ? As pointed out in the fourth section of Chapter II [33], a smaller positive constant ϑ might be selected to enhance the performances of the design. More detailed information can be found in the literature [33], we will not go into much detail here. Thus, in a later simulation we choose the parameter $\vartheta = 0.001$ and $\beta = 1$ to verify the estimation scheme.

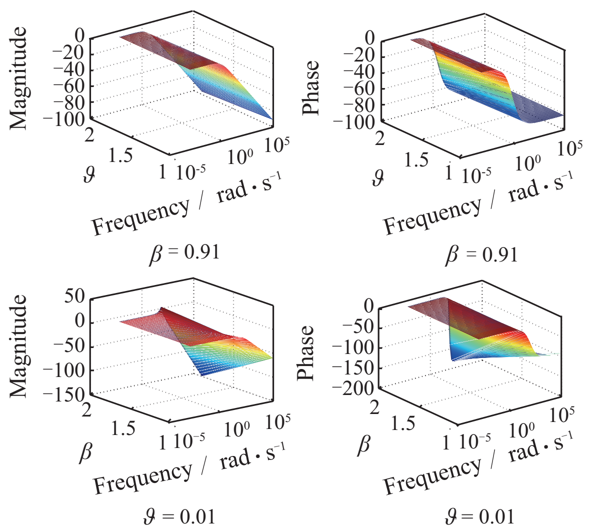


Fig. 2 Evolution of the magnitude-frequency and phase-frequency curves involving different β and ϑ

Remark 6 The RL's α-th fractional derivation of the signal $r(t)$ is itself as the order α tends to 0, while the order $\alpha \to 1$, one has

$$\begin{aligned}
&\Theta \triangleq \lim_{\alpha\to 1} {}_0^{\mathrm{RL}}D_t^{\alpha} r(t) = \\
&\lim_{\alpha\to 1} \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{r(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau = \\
&\lim_{\alpha\to 1} [\frac{r(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot r(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau] = \\
&\lim_{\alpha\to 1} [\frac{\dot r(0)t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\ddot r(\tau)}{(t-\tau)^{\alpha-1}}\mathrm{d}\tau] = \\
&\dot r(0) + \int_0^t \ddot r(\tau)\mathrm{d}\tau = \dot r(t).
\end{aligned} \tag{13}$$

On the other hand, let us consider the Laplace transform of the first expression of the equation (10)

$$\begin{aligned}
&\mathcal{L}\{\frac{\partial \phi(t,\omega)}{\partial t}\} = s\Phi(s,\omega) - \phi(0,\omega) = \\
&-\omega\Phi(s,\omega) + \mathcal{L}\{{}_0^{\mathrm{RL}}D_t^{\alpha} r(t)\} - U(s),
\end{aligned} \tag{14}$$

where $\mathcal{L}(\cdot)$ stands for the Laplace transform, moreover $\mathcal{L}\Big\{{}_0^{\mathrm{RL}}D_t^{\alpha} r(t)\Big\} = s^{\alpha}R(s) - \Big[{}_0^{\mathrm{RL}}D_t^{\alpha-1} r(t)\Big]_{t=0}$, $(0 \leqslant \alpha < 1)$, in which $\mathcal{L}(r(t)) = R(s)$; $\Phi(s,\omega)$ and $U(s)$ denote the Laplace transform of $\phi(t,\omega)$ and $u(t)$, respectively.

Note that $\phi(0,\ \omega) = e_0$, it follows from the relation (14) that

$$\Phi(s,\omega) = \frac{s^{\alpha}R(s) - [{}_0^{\mathrm{RL}}D_t^{\alpha-1} r(t)]_{t=0} - U(s) + e_0}{s+\omega}. \tag{15}$$

And the Laplace transform of the second expression of the equation (10) is

$$\begin{aligned}
&E(s) = \int_0^{\infty} \kappa(\omega)\Phi(t,\omega)\mathrm{d}\omega = \\
&\int_0^{\infty} \frac{\kappa(\omega)e_0}{s+\omega}\mathrm{d}\omega + \int_0^{\infty} \frac{\kappa(\omega)(s^{\alpha}R(s) - r_0)}{s+\omega}\mathrm{d}\omega - \\
&\int_0^{\infty} \frac{\kappa(\omega)U(s)}{s+\omega}\mathrm{d}\omega,
\end{aligned} \tag{16}$$

where $r_0 = [{}_0^{\mathrm{RL}}D_t^{\alpha-1} r(t)]_{t=0}$.

In addition, it follows from the relation (6) that

$$\int_0^{\infty} \frac{\kappa(\omega)}{s+\omega}\mathrm{d}\omega = \frac{1}{s^{\alpha}}. \tag{17}$$

This is so because

$$\mathcal{L}(\frac{t^{\alpha-1}}{\Gamma(\alpha)}) = \int_0^{\infty} \mathrm{e}^{-st}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}t = \frac{1}{s^{\alpha}},$$

and

$$\begin{aligned}
&\int_0^{\infty} \mathrm{e}^{-st}\int_0^{\infty} \kappa(\omega)\mathrm{e}^{-\omega t}\mathrm{d}\omega\mathrm{d}t = \\
&\int_0^{\infty}\int_0^{\infty} \kappa(\omega)\mathrm{e}^{-(s+\omega)t}\mathrm{d}t\mathrm{d}\omega = \int_0^{\infty} \frac{\kappa(\omega)}{s+\omega}\mathrm{d}\omega.
\end{aligned}$$

Applying the relation (17) to (16), one has

$$E(s) = \frac{1}{s^{\alpha}}\left[e_0 + s^{\alpha}R(s) - \Big[{}_0^{\mathrm{RL}}D_t^{\alpha-1} r(t)\Big]_{t=0} - U(s)\right].$$

It hence can be concluded that, when $\alpha \to 1$,

$$sE(s) = e_0 + sR(s) - r_0 - U(s). \tag{18}$$

Taking the inverse Laplace transform of both sides of the relation (18), one obtains

$$\dot e(t) = \dot r(t) - u(t). \tag{19}$$

In this situation, the Assumption 1 turns from '$\Big|{}_0^{\mathrm{RL}}D_t^{\alpha} r(t)\Big| \leqslant d_1$' into '$|\dot r(t)| \leqslant d_1$' when the order α tends to 1. Under the control law (11), it also can be shown that the closed-loop system (19)(or (10)) is asymptotically stable for any constant $k_1 \geqslant d_1$, by constructing a quadratic Lyapunov function candidate $V(t) = \frac{1}{2}e^2(t)$.

From the proof of Theorem 1, it is clear to see that the Assumption 1 is an indispensable condition. In reality, however, this upper bound is usually hard to determine and know beforehand for the RL's α-th fractional derivation of a signal. Therefore, we will be able to relax the condition of the Assumption 1; that is, the following condition is considered.

Assumption 2 For any real order $\alpha \in (0,1)$, the RL's α-th fractional derivation of $r(t)$ is an unknown function that satisfies this condition: $|{}_0^{\mathrm{RL}}D_t^{\alpha} r(t)| \leqslant \hat d_1$, where $\hat d_1$ is an unknown positive constant.

Then an adaptive scheme of fractional estimator is taken as follows:

$$\frac{\mathrm{d}k_1(t)}{\mathrm{d}t} = \dot k_1 = \hat k_1|e(t)|,\ (\hat k_1 > 0), \tag{20}$$

where $\hat k_1$ denotes the adaptive gain, the adaptive rate of $k_1(t)$ could be regulated by selection of $\hat k_1$. It is noted that $k_1(t)$ is nondecreasing function monotonously, and in general, chooses it to be nonnegative. Then this adaptive project of fractional estimator is proposed in the following theorem.

Theorem 2 Under the assumption 2, if this control law is devised as

$$u(t) = k_0 g(e(t)) + k_1(t)\mathrm{sgn}(e(t)), \tag{21}$$

where $k_1(t)$ should be to satisfy the relation (20) and, evolve to its nominal value $\hat{d}_1$. Then, the closed-loop system (10) is asymptotically stable.

Proof Consider a Lyapunov function (or energy-like function) candidate, for all $\omega \in (0, \infty)$,

$$V(t,\omega) = \frac{1}{2}\int_0^\infty \kappa(\omega)\phi^2(t,\omega)\mathrm{d}\omega + \frac{1}{2\hat{k}_1}(k_1(t) - \hat{d}_1)^2. \tag{22}$$

Then the derivative V along the trajectories of the system (10) is given by

$$\begin{aligned}\dot{V} = &-\int_0^\infty \omega\kappa(\omega)\phi^2(t,\omega)\mathrm{d}\omega + \\ &e(t)\left[{}_0^{\mathrm{RL}}D_t^\alpha r(t) - k_0 g(e(t)) - k_1(t)\mathrm{sgn}(e(t))\right] + \\ &(k_1(t) - \hat{d}_1)|e(t)| \leqslant \\ &-\int_0^\infty \omega\kappa(\omega)\phi^2(t,\omega)\mathrm{d}\omega - k_0 e(t) g(e(t)) < \\ &-k_0 e(t) g(e(t)).\end{aligned} \tag{23}$$

Therefore, one gets $\dot{V} < 0$. Similar to the rest of proof procedures of Theorem 1 and Remark 4, it can be shown that the closed-loop system (10) is asymptotically stable. □

Remark 7 Generally, the sliding mode control law usually consists of two parts: an equivalent control law and an auxiliary control law. The first one might namely be an equivalent estimation term of our previous statement $u_{\mathrm{eq}} = k_0 g(e(t)) + k_1\mathrm{sgn}(e(t))$. the other one should be a switching-type control term $u_{\mathrm{id}} = k_2|S(t)|^\gamma \mathrm{sgn}(S(t))$, in which $S(t) = {}_0I_t^{1-\alpha}e(t) + k_0\int_0^t g(e(\tau))\mathrm{d}\tau + k_1\int_0^t \mathrm{sgn}(e(\tau))\mathrm{d}\tau - {}_0I_t^{1-\alpha}r(t)$ is just the sliding manifold we constructed. When the system (10) achieves to the switching surface, it then follows that $S(t) = 0$. Further, one gets

$$\begin{aligned}\dot{S}(t) =& {}_0^{\mathrm{RL}}D_t^\alpha e(t) + k_0 g(e(t)) + k_1\mathrm{sgn}(e(t)) - \\ & {}_0^{\mathrm{RL}}D_t^\alpha r(t) = -k_2|S(t)|^{\gamma+1}.\end{aligned}$$

With $L = (1/2)S^2(t)$ as a Lyapunov function candidate, we can conclude that a reach time T which the trajectory moves inside the manifold $S(t) = 0$ satisfies the relation $T = (1/k_2(1-\gamma))|S(0)|^{1-\gamma}$, where $\gamma \neq 1$ and k_2 are real positive constant. Indeed, this switching-type estimation part in some case can be designed to meet some of the desired performance intentions including the curtailment of the reaching time and the reduction of the chattering. More detailed information can be found in the literature [34–35], we will not go into much detail here since this paper is mainly concerned with the analysis of estimation of RL's fractional derivative with the order α between 0 and 1.

Remark 8 As is known to all, there is usually a remarkable distance between the available methodologies and the potential applications. In the practical application, therefore this adaptive law (20) of our proposed estimation scheme can be further revised as follows: $\dot{k}_1(t) = \hat{k}_1|e(t)| - \delta k_1(t)$, $(\hat{k}_1 > 0)$, in which δ is a positive constant to be selected by the designer and, more detailed information can be found in the literature [36].

3.2 The case of arbitrary order

With the help of Theorem 1 or 2, the fractional estimator of RL's FO derivation of a signal $r(t)$ have be obtained for FO $\alpha \in (0,1)$. Nevertheless, fractional differentiation is mostly devoted to the research of function derivatives of arbitrary real order; that is, the order $\alpha \in \mathbb{R}^+$. Thus, how to estimate and compute RL's FO derivation of a bounded smooth linear or nonlinear signal $r(t)$ when the fractional-order α belongs to $(l-1,\ l)$, $l = 2, 3, 4\cdots$. And this will be one of the important problems to be solved in the rest of this paper. Without loss of generality, it is assumed that the arbitrary real order α can be rewritten as $m + \beta$, $(m \in \mathbb{N}, 0 \leqslant \beta < 1)$. Then, for arbitrary real order $\alpha \in \mathbb{R}^+$, the RL's α-th fractional derivation of a continuous function $x(t)$ defined on a finite interval $[0,\ b]$ could also be expressed as

$$\begin{aligned}&{}_0^{\mathrm{RL}}D_t^\alpha x(t) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}^{m+1}}{\mathrm{d}t^{m+1}}\int_0^t \frac{x(\tau)}{(t-\tau)^\beta}\mathrm{d}\tau \triangleq \\ &({}_0^{\mathrm{RL}}D_t^\beta x(t))^{(m)} = ({}_0I_t^\beta x(t))^{(m+1)} = \\ &({}_0^{\mathrm{RL}}D_t^{1-\beta}x(t))^{(m+1)},\ (t \geqslant 0),\end{aligned} \tag{24}$$

where $\alpha = m + \beta$, $(m \in \mathbb{N}, 0 \leqslant \beta < 1)$.

In order to obtain RL's fractional derivative estimator of arbitrary real order α, whose estimation model is written the following form:

$$\begin{cases}\dfrac{\partial\phi_l(t,\omega)}{\partial t} = -\omega\phi_l(t,\omega) + \\ \qquad\qquad \displaystyle\sum_{i=0}^{l-1} A_{l,i}(({}_0^{\mathrm{RL}}D_t^\beta r(t))^{(i)} - x_{l,i+1}(t)), \\ x_{l,l}(t) = u_l(t), \\ e_l(t) = \displaystyle\int_0^\infty \kappa(\omega)\phi_l(t,\omega)\mathrm{d}\omega,\end{cases} \tag{25}$$

for any $\beta \in [0,1)$. where $\phi_l(0,\omega) = e_{l0}$, $A_{l,l-1} = 1$ and $A_{l,\ i}$ is selected to meet stable condition. And we firstly need to introduce the following two assumptions.

Assumption 3 It is assumed that ${}_0^{\mathrm{RL}}D_t^\beta r(t)$, $(0 \leqslant \beta < 1)$ and its derivatives up to the $l-1$-th order are available after estimation.

Assumption 4 The RL's α-th fractional derivation of the signal $r(t)$ is an unknown fractional function that satisfies this condition: $|{}_0^{\mathrm{RL}}D_t^\alpha r(t)| = |({}_0^{\mathrm{RL}}D_t^\beta r(t))^{(l-1)}| \leqslant \hat{d}_l$, for arbitrary real order $\beta \in (0,1)$, $l = 1, 2, \cdots$, in which $\hat{d}_l$ is an unknown positive constant.

Similar to the foregoing presentation, an adaptive algorithm of fractional estimator is chosen as follows:

$$\frac{\mathrm{d}k_l(t)}{\mathrm{d}t}=\dot{k}_l=\hat{k}_l|e_l(t)|,\ (\hat{k}_l>0, l=1,2,\cdots). \quad (26)$$

Subsequently, for l-th derivative estimator of ${}_0^{\mathrm{RL}}D_t^{\beta}r(t)$ (i.e. ${}_0^{\mathrm{RL}}D_t^{\alpha}r(t)=({}_0^{\mathrm{RL}}D_t^{\beta}r(t))^{(l)}$, in which $\alpha=l+\beta, \beta\in[0,1)$), we could obtain the following corollary.

Corollary 1 Under the assumptions 3 and 4, if this control law is devised as

$$u_l(t)=k_{0l}g(e_l(t))+k_l(t)\mathrm{sgn}(e_l(t))-\sum_{i=0}^{l-2}A_{l,i}x_{l,i+1}(t), \quad (27)$$

where $k_l(t)$ should be to satisfy the relation (26) and, evolve to its nominal value $\hat{d}_l$. Then, the closed-loop system (25) is asymptotically stable.

Proof The procedure of proof follows closely that of Theorems 1 and 2, so the proof is not repeated here. Here we will only point out that $V(t,\omega)=\frac{1}{2}\int_0^{\infty}\kappa(\omega)\phi_l^2(t,\omega)\mathrm{d}\omega+\frac{1}{2\hat{k}_l}(k_l(t)-\sum_{i=0}^{l-1}A_{l,i}\hat{d}_{i+1})^2$ could be taken as a Lyapunov function (or energy-like function) candidate. Therefore, it follows that the the closed-loop system (25) is asymptotically stable. This completes the proof. □

Remark 9 The RL's fractional estimator of any higher order α is generally based on fractional estimator of lower order. For example, for the order $\alpha\in(1,2)$, how to derive its fractional estimation makes use of an available RL's fractional estimator of the order $\alpha\in(0,1)$ and, in which the boundedness of the available RL's fractional estimation is need for the signal $r(t)$.

Remark 10 When this arbitrary real order α tends to $n-1$ or $n, n\in\mathbb{N}$, one has

$$\begin{aligned}
&\varXi\triangleq\lim_{\alpha\to(n-1)^+}{}_0^{\mathrm{RL}}D_t^{\alpha}r(t)=\\
&\lim_{\alpha\to(n-1)^+}\{\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\int_0^t(t-\tau)^{n-\alpha-1}r(\tau)\mathrm{d}\tau]\}=\\
&\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\int_0^t r(\tau)\mathrm{d}\tau]=r^{(n-1)}(t),
\end{aligned}$$

and

$$\begin{aligned}
&\varPi\triangleq\lim_{\alpha\to n^-}{}_0^{\mathrm{RL}}D_t^{\alpha}r(t)=\\
&\lim_{\alpha\to n^-}\{\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\int_0^t(t-\tau)^{n-\alpha-1}r(\tau)\mathrm{d}\tau]\}=\\
&\lim_{\alpha\to n^-}[\sum_{k=0}^{n-1}\frac{r^{(k)}(0)t^{k-\alpha}}{\Gamma(k-\alpha+1)}+\frac{\int_0^t\frac{r^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau}{\Gamma(n-\alpha)}]=\\
&\lim_{\alpha\to n^-}[\frac{r^{(n)}(0)t^{n-\alpha}}{\Gamma(n-\alpha+1)}+\frac{\int_0^t\frac{r^{(n+1)}(\tau)}{(t-\tau)^{\alpha-n}}\mathrm{d}\tau}{\Gamma(n-\alpha+1)}]=\\
&r^{(n)}(0)+\int_0^t r^{(n+1)}(\tau)\mathrm{d}\tau=r^{(n)}(t).
\end{aligned}$$

Similar to the previous Remark 6, the fractional derivative estimator of the arbitrary real order α can translate into higher order derivative estimator; that is, the following model can be obtained if we want to estimate l-th derivative of a signal $r(t)$.

$$\begin{cases}\dot{e}_l(t)=\sum\limits_{i=0}^{l-1}A_{l,i}(r(t)^{(i+1)}-x_{l,\ i+1}(t)),\\ x_{l,l}(t)=u_l(t),\end{cases} \quad (28)$$

or

$$\begin{cases}e_l(t)=\sum\limits_{i=0}^{l-1}A_{l,i}(r(t)^{(i)}-x_{l,i}(t)),\\ \dot{x}_{l,\ i}(t)=x_{l,\ i+1}(t),\ (i=0,1,\cdots,l-1),\\ x_{l,l}(t)=u_l(t).\end{cases} \quad (29)$$

Obviously, under the relation (26), we can show that the closed-loop system (28) (or (29)) is asymptotically stable if the control law is chosen as $u_l(t)=k_{0l}g(e_l(t))+k_l(t)\mathrm{sgn}(e_l(t))-\sum_{i=0}^{l-2}A_{l,i}x_{l,i+1}(t)$. It here is important to note that all the derivative signals of $r(t)$ up to the $l-1$-th order are available after estimation.

Numerical simulation is one of the most important tools used to verify the rationality and feasibility of the design program. In the following section, we will give more detailed illustration involving examples with numerical simulation.

4 Illustrative example

According to the foregoing design scheme, an illustrative example is selected from two distinct functions (i.e. $\exp(-2t)$ and $3\sin(t+1)+0.5\sin(4t)$) to verify the effectiveness of our obtained results. We start by using a bounded smooth function $r(t)=\exp(-2t)$ to represent the input signal of fractional estimator, which is a luck because its fractional derivative could be accessed explicitly (i.e. ${}_0^{\mathrm{RL}}D_t^{\alpha}r(t)=t^{-\alpha}E_{1,1-\alpha}(-2t),\ t>0$ and $\alpha\in\mathbb{R}^+$, in which $E(\cdot)$ stands for Mittag-Leffler function). This is very convenient for the comparison between accurate value (AV) and estimated value (EV) of fractional derivative of the signal. When the function $g=\mathrm{e}^{5/7}$, numerical simulation of FO $\alpha=0.91$ without involving the noise is shown in Fig. 3 under the control law (11). As can be seen from the figure, the curve of EV can track the curve of AV well. Note that, following the Lemma 2, Remark 1 and the previous assumption, the value of RL fractional calculus at time $t=0$ can be nearly replaced by the value at time $t=t_0$ by choosing t_0 small enough. That is, it is supposed that their values are closely enough to each other in the numerical simulation. For instance, $|{}_0^{\mathrm{RL}}D_t^{0.91}\exp(-2t)|<5$ is obtained from $t_0=0.01$ and, $|{}_0^{\mathrm{RL}}D_t^{0.91}\exp(-2t)|<50$ is accessed from $t_0=0.001$. Incidentally, it follows from the Remark 7 that $S(0)=[{}_0I_t^{1-\alpha}e(t)]_{t=0}+k_0\int_0^0 g(e(\tau))\mathrm{d}\tau+k_1\int_0^0\mathrm{sgn}(e(\tau))\mathrm{d}\tau-[{}_0I_t^{1-\alpha}r(t)]_{t=0}=0$ (i.e. $T=0$). Therefore, there is no switching-type control when the system (9) (or (10)) is placed on the sliding mode surface. In that situation, the amplitude of vibration could be reduced during the initial phase. Now, we will use the Theorem 2 in the rest of simulation. For fractional estimator of the order $\alpha \in (0, 1]$, the simulation figures are depicted in Figs. 4–6. Figs. 4(a)–(b) show respectively that the EV of the signal can track the AV of the signal as $\alpha = 1$ and $\alpha = 0.91$, while Figs. 4(c)–(d) depict adaptive gain can converge asymptotically to constant as time progresses. From these simulation results, it is clear to see that the proposed scheme can realize tracking to the assigned function and enable the tracking error to have very good stable performance.

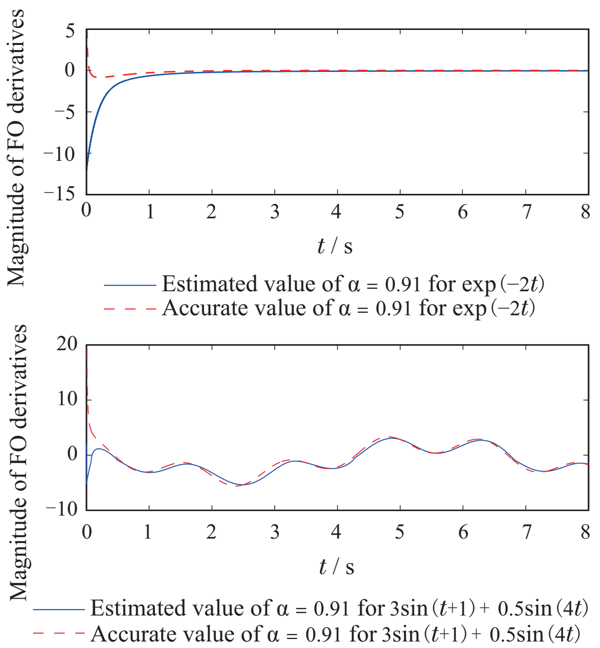
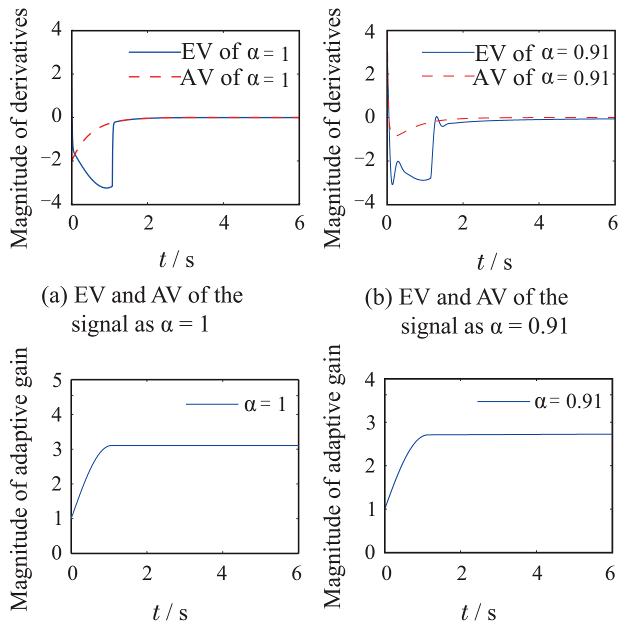


Fig. 3 Evolution of EV and AV of FO derivative without involving the noise when $k_{0_{\rm exp}} = 12$ and $k_{0_{\rm sin}} = 30$



(a) EV and AV of the signal as α = 1

(b) EV and AV of the signal as α = 0.91

(c) Adaptive gain when α = 1

(d) Adaptive gain when α = 0.91

Fig. 4 Evolution of IO and FO derivatives, and its adaptive gain for the single $\exp(-2t)$

The following simulation will illustrate further to show the superiority and effectiveness of the proposed method in the presence of a white Gaussian noise with standard deviation $\sigma = 0.01$. Taking $3\sin(t + 1) + 0.5\sin(4t)$ as the input signal of fractional estimator, its AV is derived from the Ref. [37]. Fig. 5 exhibits the relationship between AV and EV of the input single in the presence of the Gaussian noise. Meanwhile, Figs. 6(c)–(d) show separately that the adaptive law $\tilde{d}_1(t)$ with the auxiliary term $\delta\tilde{d}_1(t)$ for the FO $\alpha = 0.91$ and $\alpha = 1$ can be adjusted online by employing the available date. The corresponding Figs. 6(a)–(b) describe the varying performance of the EV and AV of the signal.

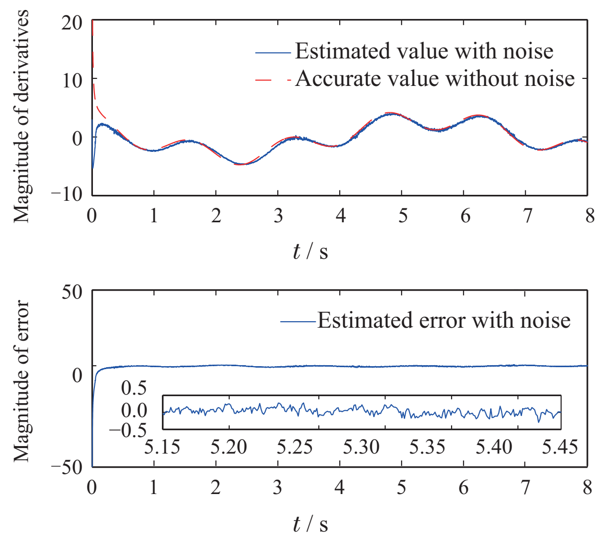
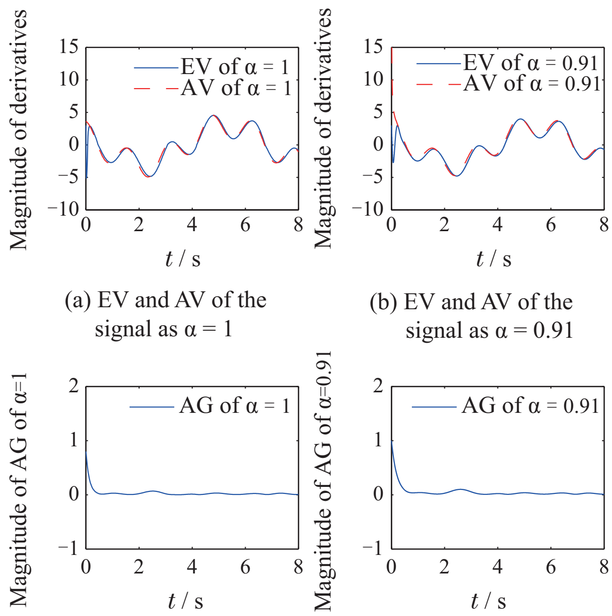


Fig. 5 Evolution of FO derivative and its error for FO $\alpha = 0.91$ when the parameter $k_{0_{\rm sin}} = 30$



(a) EV and AV of the signal as α = 1

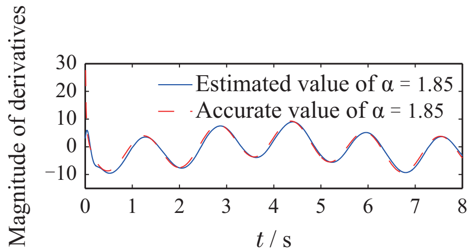
(b) EV and AV of the signal as α = 0.91

(c) Adaptive gain (AG) when α = 1

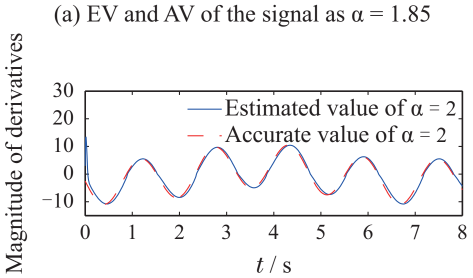
(d) Adaptive gain (AG) when α = 0.91

Fig. 6 Evolution of IO and FO derivatives, and its adaptive gain for the single $3\sin(t + 1) + 0.5\sin(4t)$ when $\delta = 8$

As for fractional-order derivative estimator of arbitrary order $\alpha \in \mathbb{R}^+$, we simulate numerically FO $\alpha = 1.85$ and integer-order $\alpha = 2$, respectively. Its figure is shown in Fig. 7. Figs. 7(a)–(b) describe respectively the varying performance of the EV and AV of the signal with order $\alpha = 1.85$ and $\alpha = 2$. As a consequence, the proposed method in this paper has quite effective performance for the estimation of RL fractional derivative of arbitrarily bounded smooth signal and, ensures to reduce some undesirable oscillations (the chattering) in the presence of noise.



(a) EV and AV of the signal as α = 1.85



(b) EV and AV of the signal as α = 2

Fig. 7 Evolution of derivatives for FO $\alpha = 1.85$ and IO $\alpha = 2$, respectively

5 Conclusion

In this paper, the problem of estimation for the RL fractional derivative of arbitrarily bounded smooth signal is presented by employing the control theoretical framework. Due to the weak singularity of fractional calculus, a novel proof about the diffusive representation of RL fractional integrals and derivatives is proposed to design and analyse the nonlinear FO derivative estimator. Based on the stability theory of the continuous frequency distributed model of RL fractional calculus, the closed-loop system of the estimated error is asymptotically stable. It is worth noting that the proposed method can not only estimate the RL fractional derivative of the signal with unknown upper bound in advance, but also adapt to the uncertain disturbance or random noise environment in the system. Finally, two functions are given to prove the validity and feasibility of the obtained results. In the future, how to embed the proposed FO estimator into the other control design and scheme will be an interesting subject.

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Appendix Performance analysis of controller

In this section, we discuss mainly the performance of the proposed controller when the signal that needs to be estimated contains random noise and uncertain disturbance. Firstly, we need to analyze the controller u via the method of the statistical linearisation. Suppose the noise of error $e(t)$ from the input signal contaminated is commonly a white Gaussian noise with standard deviation σ_e. According to the mean squared error criterion, one gets

$$J=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma_e}(u-ke)^2\exp(-\frac{e^2}{2\sigma_e^2})\mathrm{d}e,$$

where ke is assumed as the result of a statistically linearizing approximation of u. To minimize the value of J, by applying the Lagrange's method of multipliers, one has

$$k=\frac{1}{\sigma_e^2}\int_{-\infty}^{+\infty}\frac{u}{\sqrt{2\pi}\sigma_e}e\exp(-\frac{e^2}{2\sigma_e^2})\mathrm{d}e=$$
$$\frac{1}{\sigma_e^2}\int_{-\infty}^{+\infty}\frac{k_0g(e)+k_1\mathrm{sgn}(e)}{\sqrt{2\pi}\sigma_e}e\exp(-\frac{e^2}{2\sigma_e^2})\mathrm{d}e=$$
$$\underbrace{2^{\frac{q}{2p}})\sqrt{\frac{2}{\pi}}\frac{k_0\Gamma(1+\frac{q}{2p})}{\sigma_e^{1-\frac{q}{p}}}}_{\tilde{k}_0}+\underbrace{\sqrt{\frac{2}{\pi}}\frac{k_1}{\sigma_e}}_{\tilde{k}_1}, \tag{A1}$$

where $g=\mathrm{e}^{q/p}$, $p>q$ and p, q are odd. Following the structure diagram (i.e. Fig. A1) of the entire design scheme in this article, the transfer function of the AB piece for the diagram is described by

$$W(\sigma_e,s)=\frac{s^\alpha}{s^\alpha+k}.$$

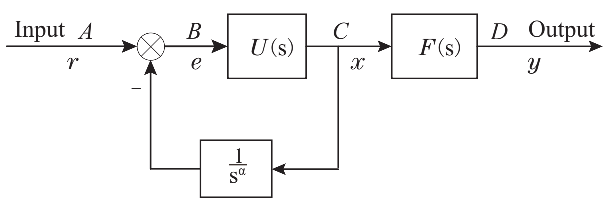

Fig. A1 A sketch diagram of estimation of fractional derivative

Setting $s=j\varpi$ results in the above relation

$$W(\sigma_e,j\varpi)=\frac{(j\varpi)^\alpha}{(j\varpi)^\alpha+k}=$$
$$\frac{\varpi^\alpha(\cos(\frac{\pi}{2}\alpha)+j\sin(\frac{\pi}{2}\alpha))}{\varpi^\alpha(\cos(\frac{\pi}{2}\alpha)+j\sin(\frac{\pi}{2}\alpha))+k}.$$

That is, the magnitude in dB is given by

$$|W(\sigma_e,j\varpi)|_{\mathrm{dB}}=20\alpha\lg(\varpi)-$$
$$20\lg((\varpi^{2\alpha}+2k\varpi^\alpha\cos(\frac{\pi}{2}\alpha)+k^2)^{\frac{1}{2}}).$$

Subsequently, a proper parameter k can be chosen such that the formula $|W(\sigma_e,j\varpi)|_{\mathrm{dB}}\approx 0$ (or $|W(\sigma_e,j\varpi)|_{\mathrm{dB}}\ll 0$) holds as $\varpi>\varpi_0$ (or $\varpi<\varpi_0$), where ϖ_0 is a value which is referred to as the separable frequency point between the useful signal and the noise signal. And, further assume that

$$\int_0^{\varpi_0}S_\zeta(\varpi)\mathrm{d}\varpi\gg\int_0^{\varpi_0}S_\xi(\varpi)\mathrm{d}\varpi,$$
$$\int_{\varpi_0}^{+\infty}S_\xi(\varpi)\mathrm{d}\varpi\gg\int_{\varpi_0}^{+\infty}S_\zeta(\varpi)\mathrm{d}\varpi,$$

where $S_\zeta(\varpi)$ and $S_\xi(\varpi)$ denote separately the spectral densities of the useful signal and the noise signal. It then is concluded that the variance of error e satisfies

$$\sigma_e^2=\mathrm{Var}(e-\mu_e)=\mathrm{Var}(e_\xi)=E(e_\xi^2)-(E(e_\xi))^2=$$
$$E(e_\xi^2)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|W(\sigma_e,j\varpi)|^2S_e(\varpi)\mathrm{d}\varpi=$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|W(\sigma_e,j\varpi)|^2(S_\zeta(\varpi)+S_\xi(\varpi))\mathrm{d}\varpi\approx$$
$$\frac{1}{2\pi}\int_{\varpi_0}^{+\infty}(S_\zeta(\varpi)+S_\xi(\varpi))\mathrm{d}\varpi\approx\frac{1}{2\pi}\int_{\varpi_0}^{+\infty}S_\xi(\varpi)\mathrm{d}\varpi=\sigma_\xi^2,$$

where $S_e(\varpi)$ represents the spectral density of error e, μ_e denotes the mathematic expectation of error e and, $\mu_e=\mu_\zeta+\mu_\xi=\mu_\zeta=e_\zeta$ in which the subscripts of character (i.e. ζ and ξ) is used to distinguish between the useful signal and the noise signal, respectively. The formula $\sigma_e\approx\sigma_\xi$ implies that the parameter k is tied up with the variance of the noise; that is, the compact correlation exists between the parameters k_0 and k_1 of Eq. (A1) and the variance σ_ξ. However, it is stressed here that the parameter k_1 which is selected should be as small as possible. The principal advantage of such selection is that the chattering phenomenon would become weaker as k_1 is chosen smaller, because the parameter k_1 is a gain of the switching term.

In addition, the transfer function of the AC piece for the Fig. A1 can be written as

$$\Phi(\sigma_e, s) = \frac{ks^\alpha}{s^\alpha + k}.$$

Again, it follows from the relationship between k and σ_e (or σ_ξ) that the formula $\Phi(\sigma_e, s) = \dfrac{s^\alpha}{\dfrac{1}{k}s^\alpha + 1} \approx s^\alpha$ when choosing relatively larger k_0 and smaller k_1 (i.e. $1/k$ should be as small as possible) in a lower frequency of noise environment. At the moment, it is equivalent to a fractional-order differentiator s^α. In turn (i.e. the higher frequency of noise environment), the relation $\Phi(\sigma_e, s) = \dfrac{k}{1 + ks^{-\alpha}} \approx k$ can be obtained. That is to say, the above device is close to a pass-through filter and skips the fractional-order differential operation. Therefore the proposed method can estimate the RL fractional derivative of the signal in the presence of the uncertain disturbance or random noise environment.

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