网络非线性系统的镇定及在车辆跟随控制的应用

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摘要: 非线性大系统或多自主体系统在理论与工程应用领域都受到了广泛的关注. 其中, 稳定性以及衍生的镇定控制问题是研究的关键. 为了应对车辆跟随控制问题, 本文针对一类下三角型不确定网络非线性系统, 给出稳定网络系统满足的充分条件, 并提出一种全局鲁棒镇定控制设计方法. 通过解决车辆跟随系统的纵向控制问题, 揭示本文的研究结果可用于输出调节问题等综合控制问题的求解. 仿真验证本文结果的有效性.

关键词: 鲁棒控制; 积分输入状态稳定; 网络系统; 非线性系统; 车辆跟随系统

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Stabilizing networked nonlinear systems and application to longitudinal platooning

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Abstract: Control of large-scale and multi-agent nonlinear systems has gained rapid developments from theory to wide engineering applications, continuously promoting more and more challenging byproduct stabilization problems. The present study is motivated by a car-following system and focuses on developing a systematic design algorithm for stabilizing a networked system with dynamic uncertainties. Our study not only gives a stabilizing control result but also explores an interesting link between the output regulation and stabilization serving a longitudinal control for a string of automated cars moving in a lane. We also show some simulation results to illustrate the proposed results.

Key words: robust control; integral input-to-state stability; networked systems; nonlinear systems; car-following systems

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1 Introduction

Feedback stabilization control is a most fundamental control topic in nonlinear control theory. Recent studies in this field have been very active for lower-triangular nonlinear systems as well as the relevant networked nonlinear systems relating to large-scale interconnected systems or networked multi-agent control systems. Particularly, such stabilization problems are essential and crucial in the synthesis of many control problems such as output regulation, synchronization, consensus, formation and others. We shall refer to [1–4] for background motivating materials on longitudinal platooning investigated in the present study. Also, we shall refer the interested readers to monographs [5–6] and references thereof on this topic and to [7–11] for a few early remarkable developments.

A breakthrough in this field can date back to the well-known backstepping technique for lower-triangular or strict-feedback nonlinear systems with free dynamic uncertainties, i.e., all the states are available for the feedback design; see [7,9] to name but a few due to our familiarity. For the more general and sophisticated circumstances such as nonlinear systems with various types of dynamic uncertainties, it has been treated by many researchers. Particularly, using the power-

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ful tools in the context of ISS (input-to-state stability), many effective stability analysis techniques as well as feedback control methodologies have been developed. For example, [10] developed a state feedback design method based on a nonlinear small-gain theorem and [12] proposed interesting Lyapunov function criterion serving stability analysis and stabilizing control for networked nonlinear systems.

Recently, as a more general stability condition than ISS, the notion of integral ISS (iISS) has been extensively studied in characterizing more general interconnected systems; see [13–15] and references therein. A very recent attempt in this direction can be [16] for studying the stabilization of nonlinear systems in the presence of iISS dynamic uncertainties. As pointed out in [16], different from the ISS dynamic uncertainties, certain bounded tolerable growth rate on system nonlinearity is almost necessary. It finally makes the stabilization problem much more challenging. In this direction, for nonlinear systems in output-feedback normal form, as a special lower-triangular nonlinear system, output-feedback design is possible. The results have been developed in [11, 17], where the former deals with either iISS\ISS or ISS dynamic uncertainties while the latter further explores the case having both iISS\ISS and ISS dynamic uncertainties. For the general lower triangular systems having multiple distinct iISS\ISS and ISS dynamic uncertainties, a recursive partial state feedback design was constructed in [18] based on a modified changing supply rate technique.

A primary objective of this paper is to investigate a stabilization problem of block lower-triangular nonlinear systems having multiple iISS dynamic uncertainties. The study is inspired by a relevant study of [16]. Specifically, by the term "iISS", we focus on a dissipation gain to be a class $\mathcal{K}$ function. Moreover, it allows the concerned dynamic uncertainties having both $\mathcal{K}\backslash\mathcal{K}_\infty$ and $\mathcal{K}_\infty$ dissipation gains. The stabilization of systems with such mixed dynamic uncertainties impacts an intermediate byproduct problem in resolving global robust cooperative output regulation by internal model-based design. This is actually a main motivation of the present stabilization study. Toward that end, the present study first presents a sufficient compact stability condition for the cascaded systems as the same one investigated in [16]. Then a systematic approach for the feedback design is developed. Compared with [16, 18], a more general class of lower-triangular systems is studied. Overall, our developed method can offer at least an interesting alternative.

Paper Organization: Section 2 presents a motivating example on a car-following system. It is used to demonstrate a byproduct but key stabilization problem. Section 3 shows the main design condition and algorithm for the stabilizing control of a class of networked nonlinear systems in lower-triangular form with multiple types of dynamic uncertainties. Section 4 gives the simulation results. Section 5 closes the paper with some remarks. All the proofs and technical details are put in Appendix.

2 Motivation example: A car-following system

This section is devoted to exploring a longitudinal control for a string of automated cars moving in a lane as shown in Figure 1. We shall re-formulate the problem as an output regulation problem. Moreover, following the idea of [19], we eventually reveal a relevant stabilization problem as an important step to manage this longitudinal platooning control problem.

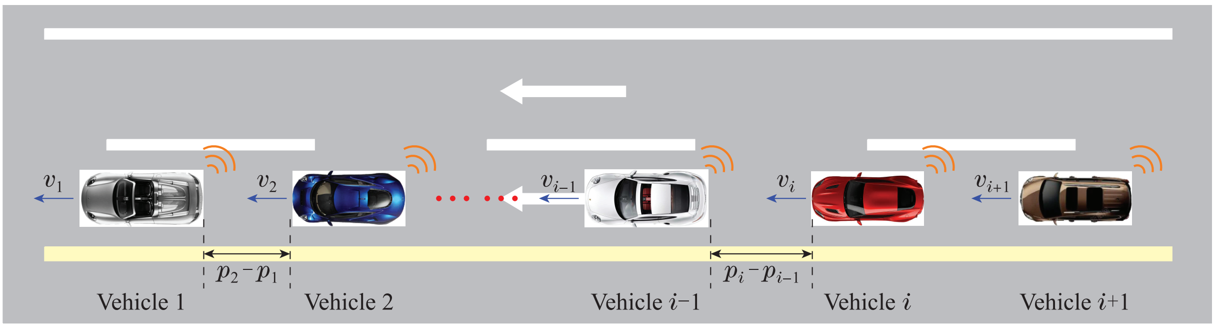

Fig. 1 Car following within an automated lane

Specifically, we focus on a working example of a car-following system adopted from [20] and described by

$$\begin{cases} \dot{\psi}_i = v_i - v_{i-1}, \\ \dot{v}_i = \dfrac{1}{m_i}(-A_{\rho i}v_i^2 - d_i + f_i), \\ \dot{f}_i = \dfrac{1}{\tau_i}(-f_i + u_i), \end{cases} \tag{1}$$

for $i = 1, 2, \cdots, N$, where v_0 is the velocity of a virtual leader specified later in (2), and $\psi_i = p_i - p_{i-1}$ is the relative distance between the ith and $i-1$st vehicles. The meaning of other symbols are listed in Table 1. Roughly speaking, the longitudinal controller is drive all vehicles to maintain a steady-state velocity with vehicle-to-vehicle spacing constraints, and meanwhile to follow a leader vehicle at a safe distance. For case studies, we shall refer to [4, 21] and relevant references therein for more interesting circumstances.

Table 1 Vehicle variables and parameters

Symbol	Meaning	Nominal value (of the ith vehicle)
m_i	Mass	130 kg
$A_{\rho i}$	Aerodynamic drag coefficient	0.3 $\mathrm{Ns^2/m^2}$
d_i	Constant frictional force	10 N
τ_i	Engine time constant	0.2 s
v_i	Vehicle velocity	–
f_i	Actuator force applied to the vehicle	–
u_i	Control input	–

Associated with (1), a *virtual* leader vehicle is set whose motion satisfies

$$\dot{p}_0 = v_0,\ v_0 > 0. \tag{2}$$

Here, p_0 and v_0 are the lead vehicle's position and velocity, respectively. We assume that the lead vehicle's velocity is a time-varying sinusoidal signal but with a single frequency as follows:

$$v_0(t) = A_1\sin(\omega t + \phi_1) + A_0, \tag{3}$$

where A_1, ω, ϕ_1, A_0 are real parameters with

$$A_1, \omega, A_0 > 0 \text{ and } A_0 \gg A_1.$$

For the purpose of modeling the reference and uncertain parameters, we define the following two exosystems:

$$\begin{cases} \dot{w}_1 = S_1 w_1, \\ p_{01} = Q_1 w_1 \end{cases} \text{and} \begin{cases} \dot{w}_2 = S_2(\omega) w_2, \\ p_{02} = Q_2 w_2, \end{cases} \tag{4}$$

where

$$S_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ S_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},$$

for some matrices Q_1 and Q_2.

We assume that $\omega \in \boldsymbol{S} \subset \mathbb{R}$, $w_1(0) \in \boldsymbol{W}_1 \subset \mathbb{R}^2$, and $w_2(0) \in \boldsymbol{W}_2 \subset \mathbb{R}^2$ with $\boldsymbol{S}$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ being any known compact sets.

For the ease of presentation, we denote

$$\begin{aligned} & w = [w_1^{\mathrm{T}}\ \ w_2^{\mathrm{T}}]^{\mathrm{T}},\ \mu = [\mu_1^{\mathrm{T}}\ \ \cdots\ \ \mu_N^{\mathrm{T}}\ \ w^{\mathrm{T}}]^{\mathrm{T}}, \\ & \mu_i = [m_i\ \ A_{\rho i}\ \ d_i\ \ \tau_i]^{\mathrm{T}}, \\ & Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix},\ S(\omega) = \begin{bmatrix} S_1 & 0 \\ 0 & S_2(\omega) \end{bmatrix}. \end{aligned}$$

Thus, the exosystems in (4) can be written in a compact as

$$\begin{cases} \dot{w} = S(\omega) w, \\ p_0 = Q w. \end{cases} \tag{5}$$

Consequently, the lead vehicle's velocity (3) satisfies

$$v_0 = QS(\omega)w.$$

In what follows, we denote

$$(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}) := (p_i, v_i, f_i, u_i),\ i = 1, \cdots, N.$$

System (1) can be written as

$$\begin{cases} \dot{x}_{i,1} = x_{i,2}, \\ \dot{x}_{i,2} = \dfrac{1}{m_i} x_{i,3} + \dfrac{1}{m_i}(-A_{\rho i} x_{i,2}^2 - d_i), \\ \dot{x}_{i,3} = \dfrac{1}{\tau_i} x_{i,4} - \dfrac{1}{\tau_i} x_{i,3}. \end{cases} \tag{6}$$

Then the absolute position tracking error is given by

$$e_i = x_{i,1} - x_{i,0} + \bar{L}_i,\ i = 1, \cdots, N,$$

where $x_{i,0} = p_0$ is the lead vehicle's position and $\bar{L}_i = \sum\limits_{k=1}^{i} L_k$ with $L_k > 0$ being the desired constant inter-vehicle spacing. For the purpose of longitudinal control, the inner relative distance error between the ith and $i-1$st vehicles defined is

$$\widehat{e}_i = \psi_i + L_i = x_{i,1} - x_{i-1,i} + L_i,\ i = 1, \cdots, N.$$

Now we formulate the control goal of the car-following system. The objective is to develop a controller for system (1) such that for each initial conditions the trajectory of closed-loop system exists for all $t \geqslant 0$, and the regulated output satisfies $\lim\limits_{t\to\infty} e(t) = 0$.

For this purpose, we follow the two-step design procedure for solving output regulation problems in the light of [22] coming up with a key stabilization problem. To this end, first, an internal model candidate is constructed to make compensation for steady-state input. Then, the output regulation problem of the original system is converted into a stabilization problem of an augmented system composed of the system dynamics and the internal model.

Denote the steady-state state of $x_{i,1}$, $x_{i,2}$, $x_{i,3}$ and $x_{i,4}$ by

$$\begin{aligned} & x_{i,1}^* := x_{i,1}^*(\mu),\ x_{i,2}^* := x_{i,2}^*(\mu), \\ & x_{i,3}^* := x_{i,3}^*(\mu),\ x_{i,4}^* := x_{i,4}^*(\mu), \end{aligned}$$

respectively. Then, by (6) and (5), we have

$$\begin{cases} \dot{x}_{i,1}^* = x_{i,2}^*, \\ \dot{x}_{i,2}^* = \dfrac{1}{m_i} x_{i,3}^* + \dfrac{1}{m_i}(-A_{\rho i} x_{i,2}^{*2} - d_i), \\ \dot{x}_{i,3}^* = \dfrac{1}{\tau_i} x_{i,4}^* - \dfrac{1}{\tau_i} x_{i,3}^*, \\ 0 = x_{i,1}^* - x_{i,0}(\mu) + \bar{L}_i. \end{cases} \tag{7}$$

Further solving the above regulator equations gives the following steady-state states and inputs

$$x_{i,1}^* = x_{i,0}(\mu) - \bar{L}_i,\ x_{i,0}(\mu) = Q\bar{v},$$

$$x_{i,2}^* = \frac{\partial x_{i,1}^*}{\partial \bar{v}} S(\omega)\bar{v},$$

$$x_{i,3}^* = m_i \frac{\partial x_{i,2}^*}{\partial \bar{v}} S(\omega)\bar{v} + A_{\rho i} x_{i,2}^{*2} + d_i,$$

$$x_{i,4}^* = \tau_i \frac{\partial x_{i,3}^*}{\partial \bar{v}} S(\omega)\bar{v} + x_{i,3}^*.$$

Note that functions $x_{i,j}^*$, $i = 1, \cdots, N$, $j = 1, \cdots, 4$ in question are all polynomials in their arguments. It allows us to apply the nonlinear internal model design method as in [23] to manage the internal model design.

To this end, for each $i = 1, \cdots, N$, $j = 1, 2, 3$, we first choose a controllable pair $(M_{i,j}^\circ, N_{i,j}^\circ)$ of the form

$$M_{i,j}^\circ = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -m_{i,j,1} & -m_{i,j,2} & \cdots & -m_{i,j,s_{ij}} \end{bmatrix},$$

$$N_{i,j}^\circ = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

for some positive integer s_{ij}, with

$$m_{i,j} = \begin{bmatrix} m_{i,j,1} & m_{i,j,2} & \cdots & m_{i,j,s_{ij}} \end{bmatrix}^{\mathrm{T}},$$

chosen such that $M_{i,j}^\circ$ is Hurwitz.

To handle the uncertain exosystem (5), we adopt internal model candidates as follows, for $i = 1, \cdots, N$, $j = 1, 2, 3$,

$$\begin{cases} \dot{\eta}_{i,j}^{a1} = M_{i,j}^\circ \eta_{i,j}^{a1} + N_{i,j}^\circ \eta_{i,j}^{a2}, \\ \dot{\eta}_{i,j}^{a2} = -\eta_{i,j}^{a2} + x_{i,j+1}, \\ \dot{\eta}_{i,j}^{b} = -\eta_{i,j}^{a1} \left[(\eta_{i,j}^{a1})^{\mathrm{T}} \eta_{i,j}^{b} - \eta_{i,j}^{a2} \right], \end{cases} \tag{8}$$

or equivalently, written in the following compact form

$$\begin{cases} \dot{\eta}_{i,j}^{a} = \gamma_{i,j}^{a}(\eta_{i,j}^{a}) + N_{i,j} x_{i,j+1}, \\ \dot{\eta}_{i,j}^{b} = \gamma_{i,j}^{b}(\eta_{i,j}^{a}, \eta_{i,j}^{b}), \end{cases} \tag{9}$$

with output $x_{i,j+1}$, where

$$\eta_{i,j}^a = \mathrm{col}(\eta_{i,j}^{a1}, \eta_{i,j}^{a2}),\ \gamma_{i,j}^a(\eta_{i,j}^a) = M_{i,j}\eta_{i,j}^a,$$

$$M_{i,j} = \begin{bmatrix} M_{i,j}^\circ & N_{i,j}^\circ \\ 0 & 1 \end{bmatrix},\ N_{i,j} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Denote the steady-state states of $\eta_{i,j}^a$ and $\eta_{i,j}^b$ as $\theta_{i,j}^a := \theta_{i,j}^a(\mu)$ and $\theta_{i,j}^b := \theta_{i,j}^a(\mu)$, respectively. Then, from (9), we have

$$\dot{\theta}_{i,j}^a = \gamma_{i,j}^a(\theta_{i,j}^a) + N_{i,j} x_{i,j+1}^*,$$
$$\dot{\theta}_{i,j}^b = \gamma_{i,j}^b(\theta_{i,j}^a, \theta_{i,j}^b),$$

for $i = 1, \cdots, N$, $j = 1, 2, 3$. Further, the steady-state state and input can be expressed as[1]

$$x_{i,j+1}^*(\mu) = \Gamma_{i,j}(\theta_{i,j}^a(\mu), \theta_{i,j}^b(\mu)). \tag{10}$$

By attaching internal model (9) to (6), we obtain the following augmented system, for $i = 1, \cdots, N$,

$$\begin{cases} \dot{\eta}_{i,1}^a = \gamma_{i,1}^a(\eta_{i,1}^a) + N_{i,1} x_{i,2}, \\ \dot{\eta}_{i,1}^b = \gamma_{i,1}^b(\eta_{i,1}^a, \eta_{i,1}^b), \\ \dot{x}_{i,1} = x_{i,2}, \end{cases} \tag{11a}$$

$$\begin{cases} \dot{\eta}_{i,2}^a = \gamma_{i,2}^a(\eta_{i,2}^a) + N_{i,2} x_{i,3}, \\ \dot{\eta}_{i,2}^b = \gamma_{i,2}^b(\eta_{i,2}^a, \eta_{i,2}^b), \\ \dot{x}_{i,2} = \dfrac{1}{m_i} x_{i,3} + \dfrac{1}{m_i}(-A_{\rho i} x_{i,2}^2 - d_i), \end{cases} \tag{11b}$$

$$\begin{cases} \dot{\eta}_{i,3}^a = \gamma_{i,3}^a(\eta_{i,3}^a) + N_{i,3} x_{i,4}, \\ \dot{\eta}_{i,3}^b = \gamma_{i,3}^b(\eta_{i,3}^a, \eta_{i,3}^b), \\ \dot{x}_{i,3} = \dfrac{1}{\tau_i} x_{i,4} - \dfrac{1}{\tau_i} x_{i,3}. \end{cases} \tag{11c}$$

Define the following new coordinates and input transformations for $i = 1, \cdots, N$,

$$\begin{cases} e_i = x_{i,1} - x_{i,1}^*,\ \bar{x}_{i,1} = e_i, \\ \bar{x}_{i,j+1} = x_{i,j+1} - \Gamma_{i,j}(\eta_{i,j}^a, \eta_{i,j}^b),\ j = 1, 2, 3, \\ \tilde{\eta}_{i,1}^a = \eta_{i,1}^a - \theta_{i,1}^a - N_{i,1} e_i, \\ \tilde{\eta}_{i,2}^a = \eta_{i,2}^a - \theta_{i,2}^a - m_i N_{i,2} \bar{x}_{i,2}, \\ \tilde{\eta}_{i,3}^a = \eta_{i,3}^a - \theta_{i,3}^a - \tau_i N_{i,3} \bar{x}_{i,3}, \\ \tilde{\eta}_{i,j}^b = \eta_{i,j}^b - \theta_{i,j}^b,\ j = 1, 2, 3. \end{cases} \tag{12}$$

It gives a translated augmented system described by the following equations:

$$\begin{cases} \dot{\tilde{\eta}}_{i,1}^a = M_{i,1}\tilde{\eta}_{i,1}^a + M_{i,1} N_{i,1} e_i, \\ \dot{\tilde{\eta}}_{i,1}^b = -\Theta_{i,1}(\mu)\tilde{\eta}_{i,1}^b + \varphi_{i,1}^b(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \mu), \\ \dot{e}_i = \bar{x}_{i,2} + \Delta_{i,1}(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \mu), \end{cases} \tag{13a}$$

$$\begin{cases} \dot{\tilde{\eta}}_{i,2}^a = M_{i,2}\tilde{\eta}_{i,2}^a + \varphi_{i,2}^a(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \bar{x}_{i,2}, \mu), \\ \dot{\tilde{\eta}}_{i,2}^b = -\Theta_{i,2}(\mu)\tilde{\eta}_{i,2}^b + \varphi_{i,2}^b(\tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, \bar{x}_{i,2}, \mu), \\ \dot{\bar{x}}_{i,2} = \dfrac{1}{m_i}\bar{x}_{i,3} + \Delta_{i,2}(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, \tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, e_i, \bar{x}_{i,2}, \mu), \end{cases} \tag{13b}$$

$$\begin{cases} \dot{\tilde{\eta}}_{i,3}^a = M_{i,3}\tilde{\eta}_{i,3}^a + \varphi_{i,3}^a(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, \tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, e_i, \bar{x}_{i,2}, \\ \qquad \bar{x}_{i,3}, \mu), \\ \dot{\tilde{\eta}}_{i,3}^b = -\Theta_{i,3}(\mu)\tilde{\eta}_{i,3}^b + \varphi_{i,3}^b(\tilde{\eta}_{i,3}^a, \tilde{\eta}_{i,3}^b, \bar{x}_{i,3}, \mu), \\ \dot{\bar{x}}_{i,3} = \dfrac{1}{\tau_i}\bar{x}_{i,4} + \Delta_{i,3}(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, \tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, \tilde{\eta}_{i,3}^a, \tilde{\eta}_{i,3}^b, \\ \qquad e_i, \bar{x}_{i,2}, \bar{x}_{i,3}, \mu), \end{cases} \tag{13c}$$

The explicit expression of the above function is as follows:

[1] The functions $\Gamma_{i,j}(\cdot)$, $i = 1, \cdots, N$, $j = 1, 2, 3$ can be chosen smooth and compactly supported.

$$\begin{cases} \varTheta_{i,1}^{a} = \theta_{i,1}^{a1}(\mu)[\theta_{i,1}^{a1}(\mu)]^{\mathrm{T}},\ \varTheta_{i,2}^{a} = \theta_{i,2}^{a1}(\mu)[\theta_{i,2}^{a1}(\mu)]^{\mathrm{T}},\ \varTheta_{i,3}^{a} = \theta_{i,3}^{a1}(\mu)[\theta_{i,3}^{a1}(\mu)]^{\mathrm{T}}, \\ \varphi_{i,1}^{b} = -(\tilde{\eta}_{i,1}^{a1} + \theta_{i,1}^{a1})[(\tilde{\eta}_{i,1}^{a1} + \theta_{i,1}^{a1})^{\mathrm{T}}\theta_{i,1}^{b} - \tilde{\eta}_{i,1}^{a2} - \theta_{i,1}^{a2} - e_i] - [\tilde{\eta}_{i,1}^{a1}\tilde{\eta}_{i,1}^{a1\mathrm{T}} + \tilde{\eta}_{i,1}^{a1}\theta_{i,1}^{a1\mathrm{T}} + \theta_{i,1}^{a1}\tilde{\eta}_{i,1}^{a1\mathrm{T}}]\theta_{i,1}^{b} + \\ \qquad \theta_{i,1}^{a1}[\theta_{i,1}^{a1\mathrm{T}}\theta_{i,1}^{b} - \theta_{i,1}^{a2}], \\ \varDelta_{i,1} = \tilde{\varGamma}_{i,1}(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, e_i, \mu), \\ \varphi_{i,2}^{a} = m_i M_{i,2} N_{i,2}\bar{x}_{i,2} + N_{i,2}\rho_{i,2}(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, e_i, \bar{x}_{i,2}, \mu) - N_{i,2}\tilde{\varGamma}_{i,1}'(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, e_i, \bar{x}_{i,2}, \mu), \\ \varphi_{i,2}^{b} = -[\tilde{\eta}_{i,2}^{a1} + \theta_{i,2}^{a1}][(\tilde{\eta}_{i,2}^{a1} + \theta_{i,2}^{a1})^{\mathrm{T}}\theta_{i,2}^{b} - \tilde{\eta}_{i,2}^{a2} - \theta_{i,2}^{a2} - m_i x_{i,2}] - [\tilde{\eta}_{i,2}^{a1}\tilde{\eta}_{i,2}^{a1\mathrm{T}} + \tilde{\eta}_{i,2}^{a1}\theta_{i,2}^{a1\mathrm{T}} + \theta_{i,2}^{a1}\tilde{\eta}_{i,2}^{a1\mathrm{T}}]\theta_{i,2}^{b} + \\ \qquad \theta_{i,2}^{a1}[\theta_{i,2}^{a1\mathrm{T}}\theta_{i,2}^{b} - \theta_{i,2}^{a2}], \\ \varDelta_{i,2} = \dfrac{1}{m_i}\tilde{\varGamma}_{i,2}(\tilde{\eta}_{i,2}^{a}, \tilde{\eta}_{i,2}^{b}, \bar{x}_{i,2}, \mu) - \tilde{\varGamma}_{i,1}'(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, e_i, \bar{x}_{i,2}, \mu) + \rho_{i,2}(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, e_i, \bar{x}_{i,2}, \mu), \\ \varphi_{i,3}^{a} = \tau_i M_{i,3} N_{i,3} x_{i,3} + N_{i,3}\tilde{\varGamma}_{i,2}'(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, \tilde{\eta}_{i,2}^{a}, \tilde{\eta}_{i,2}^{b}, e_i, \bar{x}_{i,2}, \mu) - N_{i,3}\rho_{i,3}(\tilde{\eta}_{i,2}^{a}, \tilde{\eta}_{i,2}^{b}, \bar{x}_{i,2}, \bar{x}_{i,3}, \mu), \\ \varphi_{i,3}^{b} = -[\tilde{\eta}_{i,3}^{a1} + \theta_{i,3}^{a1}][(\tilde{\eta}_{i,3}^{a1} + \theta_{i,3}^{a1})^{\mathrm{T}}\theta_{i,3}^{b} - \tilde{\eta}_{i,3}^{a2} - \theta_{i,3}^{a2} - \tau_i x_{i,3}] - [\tilde{\eta}_{i,3}^{a1}\tilde{\eta}_{i,3}^{a1\mathrm{T}} + \tilde{\eta}_{i,3}^{a1}\theta_{i,3}^{a1\mathrm{T}} + \theta_{i,3}^{a1}\tilde{\eta}_{i,3}^{a1\mathrm{T}}]\theta_{i,3}^{b} + \\ \qquad \theta_{i,3}^{a1}[\theta_{i,3}^{a1\mathrm{T}}\theta_{i,3}^{b} - \theta_{i,3}^{a2}], \\ \varDelta_{i,3} = \dfrac{1}{\tau_i}\tilde{\varGamma}_{i,3}(\tilde{\eta}_{i,3}^{a}, \tilde{\eta}_{i,3}^{b}, \bar{x}_{i,3}, \mu) - \tilde{\varGamma}_{i,2}'(\tilde{\eta}_{i,1}^{a}, \tilde{\eta}_{i,1}^{b}, \tilde{\eta}_{i,2}^{a}, \tilde{\eta}_{i,2}^{b}, e_i, \bar{x}_{i,2}, \mu) + \rho_{i,3}(\tilde{\eta}_{i,2}^{a}, \tilde{\eta}_{i,2}^{b}, \bar{x}_{i,2}, \bar{x}_{i,3}, \mu), \\ \tilde{\varGamma}_{i,1} = \varGamma_{i,1}(\tilde{\eta}_{i,1}^{a} + \theta_{i,1}^{a} + N_{i,1}e_i, \tilde{\eta}_{i,1}^{b} + \theta_{i,1}^{b}) - \varGamma_{i,1}(\theta_{i,1}^{a}, \theta_{i,1}^{b}), \\ \tilde{\varGamma}_{i,2} = \varGamma_{i,2}(\tilde{\eta}_{i,2}^{a} + \theta_{i,2}^{a} + m_i N_{i,2}\bar{x}_{i,2}, \tilde{\eta}_{i,2}^{b} + \theta_{i,2}^{b}) - \varGamma_{i,2}(\theta_{i,2}^{a}, \theta_{i,2}^{b}), \\ \tilde{\varGamma}_{i,3} = \varGamma_{i,3}(\tilde{\eta}_{i,3}^{a} + \theta_{i,3}^{a} + \tau_i N_{i,3}\bar{x}_{i,3}, \tilde{\eta}_{i,3}^{b} + \theta_{i,3}^{b}) - \varGamma_{i,3}(\theta_{i,3}^{a}, \theta_{i,3}^{b}), \\ \tilde{\varGamma}_{i,1}' = \dfrac{\partial \varGamma_{i,1}(\eta_{i,1}^{a}, \eta_{i,1}^{b})}{\partial \eta_{i,1}^{a}}(\dot{\tilde{\eta}}_{i,1}^{a} + \dot{\theta}_{i,1}^{a} + N_{i,1}\dot{e}_i) + \dfrac{\partial \varGamma_{i,1}(\eta_{i,1}^{a}, \eta_{i,1}^{b})}{\partial \eta_{i,1}^{b}}(\dot{\tilde{\eta}}_{i,1}^{b} + \dot{\theta}_{i,1}^{b}) - \dfrac{\mathrm{d}\varGamma_{i,1}(\theta_{i,1}^{a}, \theta_{i,1}^{b})}{\mathrm{d}t}, \\ \tilde{\varGamma}_{i,2}' = \dfrac{\partial \varGamma_{i,2}(\eta_{i,2}^{a}, \eta_{i,2}^{b})}{\partial \eta_{i,2}^{a}}(\dot{\tilde{\eta}}_{i,2}^{a} + \dot{\theta}_{i,2}^{a} + m_i N_{i,2}\dot{\bar{x}}_{i,2}) + \dfrac{\partial \varGamma_{i,2}(\eta_{i,2}^{a}, \eta_{i,2}^{b})}{\partial \eta_{i,2}^{b}}(\dot{\tilde{\eta}}_{i,2}^{b} + \dot{\theta}_{i,2}^{b}) - \dfrac{\mathrm{d}\varGamma_{i,2}(\theta_{i,2}^{a}, \theta_{i,2}^{b})}{\mathrm{d}t}, \\ \rho_{i,2} = \dfrac{A_{\rho i}}{m_i}[x_{i,3}^{*2} - \bar{x}_{i,2} - \big(\varGamma_{i,1}(\tilde{\eta}_{i,1}^{a} + \theta_{i,1}^{a} + N_{i,1}e_i, \tilde{\eta}_{i,1}^{b} + \theta_{i,1}^{b})\big)^2], \\ \rho_{i,3} = \dfrac{1}{\tau_i}[x_{i,3}^{*} - \bar{x}_{i,3} - \varGamma_{i,2}(\tilde{\eta}_{i,2}^{a} + \theta_{i,2}^{a} + m_i N_{i,2}\bar{x}_{i,2}, \tilde{\eta}_{i,2}^{b} + \theta_{i,2}^{b})]. \end{cases} \tag{14}$$

A detailed calculation for deriving the translated augmented system (13) can be found in Appendix A.

At this place, we note that the global robust output regulation problem for system (6) and (5) is now converted into an important global robust stabilization problem for the translated augmented system (13). The latter stabilizing control of (13) is the focus of the present study. It motivates the study in the next section.

3 Stabilization for setting an iISS network

This section is to carry out a global robust partial-state feedback stabilization design for a general lower-triangular nonlinear system that can be viewed as byproduct systems in internal model based approach for solving the cooperative output regulation problem.

Specifically, summarized from (13) in preceding motivating example, the networked nonlinear system at issue is given by

$$\begin{cases} \dot{z}_i = f_i^a(z_{[i]}, \zeta_{[i]}, x_{[i]}, \mu), \\ \dot{\zeta}_i = f_i^b(z_{[i]}, \zeta_{[i]}, x_{[i]}, \mu), \\ \dot{x}_i = H_i x_{i+1} + f_i^c(z_{[i]}, \zeta_{[i]}, x_{[i]}, \mu),\ 1 \leqslant i \leqslant n, \end{cases} \tag{15}$$

where $x_i := [x_{i,1}\ \cdots\ x_{i,N}]^{\mathrm{T}} \in \mathbb{R}^N$ is the partial measured state for $1 \leqslant i \leqslant n$, $x_{n+1} := [u_1\ \cdots\ u_N] \in \mathbb{R}^N$ is the control. Both $z_i \in \mathbb{R}^{n_{zi}}$ and $\zeta_i \in \mathbb{R}^{n_{\zeta i}}$ are dynamic uncertainties and $\mu := \mu(t) \in \boldsymbol{D}$ is static uncertainty that continuously varies in a compact set $\boldsymbol{D}$. All functions f_i^a, f_i^b, f_i^c are assumed to be sufficiently smooth with $f_i^a(0,0,0,\mu) = 0$, $f_i^b(0,0,0,\mu) = 0$, $f_i^c(0,0,0,\mu) = 0$ for $1 \leqslant i \leqslant n$. The matrix H_i is assumed to be positive definite and can be related to the Laplacian matrix of the communication topology in cooperative output regulation such as [23–24] for instance.

As motivated in the preceding section, the main control goal in the present study is to design a decentralized partial-state feedback controller

$$u_l = \kappa_l(x_{1,l}, \cdots, x_{n,l}),\ 1 \leqslant l \leqslant N, \tag{16}$$

for designing functions $\kappa_l : \mathbb{R}^n \to \mathbb{R}$, such that the closed-loop system composed of (15) and (16) is globally robustly asymptotically stable at the origin $(z_i, \zeta_i, x_i) = (0, 0, 0), 1 \leqslant i \leqslant n$.

The basic idea for tacking the aforementioned global robust stabilization problem is to first establish a stability condition for a typical class of iISS networks and then pursue the partial-state feedback design fulfilling the stability condition. These will be elaborated in following two subsections, respetively.

3.1 A sufficient stability condition

In this subsection, we will present a set of verifiable conditions for the following decomposition network with $m = 3n$,

$$\dot{\chi}_k = f_k(\chi_1, \chi_2, \cdots, \chi_m, \mu),\ 1 \leqslant k \leqslant m, \tag{17}$$

with $\chi_k \in \mathbb{R}^{n_k}$ and each function f_k being sufficiently smooth and $f_k(0, 0, \cdots, 0, \mu) = 0$ for $1 \leqslant i \leqslant m$.

For the sake of convenience, denote

$$\boldsymbol{n}_a := \{3i - 2 : 1 \leqslant i \leqslant n\},$$
$$\boldsymbol{n}_b := \{3i - 1 : 1 \leqslant i \leqslant n\},$$
$$\boldsymbol{n}_c := \{3i : 1 \leqslant i \leqslant n\}.$$

Assumption 1 For the network (17), there exist iISS Lyapunov functions $\{V_k := V_k(t, \chi_k)\}_{k=1}^m$ satisfying along trajectories of (17),

$$\begin{cases} \underline{\alpha}_k(\|\chi_k\|) \leqslant V_k(t, \chi_k) \leqslant \bar{\alpha}_k(\|\chi_k\|), \\ \dot{V}_k \leqslant \sum\limits_{l=1}^{m} \gamma_{k,l}(V_l),\ \gamma_{k,k}(V_i) = -\alpha_k(V_k), \end{cases} \tag{18}$$

where $\underline{\alpha}_k, \bar{\alpha}_k \in \mathcal{K}_\infty$ and[2]

$$\gamma_{l,k} \in \mathcal{K} \cap \mathcal{O}(\alpha_k),\ \alpha_k \in \mathcal{K}_\infty,$$
$$\text{for } (k, l) : k \in \boldsymbol{n}_a \text{ and } k < l \leqslant m, \tag{19a}$$
$$\gamma_{l,k} \in \mathcal{K}^o \cap \mathcal{O}(\alpha_k),\ \ \alpha_k \in \mathcal{K}^o,$$
$$\text{for } (k, l) : k \in \boldsymbol{n}_b \text{ and } k < l \leqslant m, \tag{19b}$$
$$\gamma_{l,k} \in \mathcal{K} \cap \mathcal{O}(Id),$$
$$\text{for } (k, l) : i \in \boldsymbol{n}_c \text{ and } k - 3 \leqslant l \leqslant m, \tag{19c}$$
$$\gamma_{l,k} \equiv 0,\ \text{for } (k, l) : k \in \boldsymbol{n}_a \cup \boldsymbol{n}_b \text{ and } 1 \leqslant l < k, \tag{19d}$$
$$\gamma_{l,k} \equiv 0,\ \text{for } (k, l) : k \in \boldsymbol{n}_c \text{ and } 1 \leqslant l < k - 3, \tag{19e}$$

For an intuitive illustration of Assumption 1, we can define a block square matrix in term of the functions $\gamma_{l,k}$ as

$$\left(\begin{array}{ccc|ccc|c} \alpha_1 & 0 & \gamma_{1,3} & 0 & 0 & 0 & \cdots \\ \gamma_{2,1} & \alpha_2 & \gamma_{2,3} & 0 & 0 & 0 & \cdots \\ \gamma_{3,1} & \gamma_{3,2} & \alpha_3 & 0 & 0 & \gamma_{3,6} & \cdots \\ \hline \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,3} & \alpha_4 & 0 & \gamma_{4,6} & \cdots \\ \gamma_{5,1} & \gamma_{5,2} & \gamma_{5,3} & \gamma_{5,4} & \alpha_5 & \gamma_{5,6} & \cdots \\ \gamma_{6,1} & \gamma_{6,2} & \gamma_{6,3} & \gamma_{6,4} & \gamma_{6,5} & \alpha_6 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \tag{20}$$

with respect to the following block partitions from the first block (χ_1, χ_2, χ_3) to the nth block $(\chi_{m-2}, \chi_{m-1}, \chi_m)$ of (17). In the matrix (20), functions α_k's are imposed in the diagonal entries and in each column corresponding to the index in $\boldsymbol{n}_a \cup \boldsymbol{n}_b$, all off-diagonal entries are necessary to be the same type of functions with the diagonal one. Moreover, conditions in (19c), (19d), and (19e) are introduced to assure existence of $\alpha_k \in \mathcal{K}, k \in \boldsymbol{n}_c$.

Stability and stabilization problems relating to ISS networked nonlinear systems have been extensively studied in literature, see [5, 10] for an excellent overview. In sharp contrast to that, as shown in the preceding section, in many situations, the networked systems can have the more general iISS dynamic uncertainties. In fact, a pioneering work is [12] where a network stability criterion is proposed for a general iISS network. Based on that, we further establish the following useful stability result for system (17) as the design criterion.

Lemma 1 Consider the iISS network (17) with $m = 3n$ under Assumption 1. One can construct some gain functions $\alpha_k \in \mathcal{K} \cap \mathcal{O}(Id)$ for $k \in \boldsymbol{n}_c$ such that the rendering the network (17) is globally asymptotically stable at $\chi_k = 0$ for $1 \leqslant k \leqslant m$.

Lemma 1 can be viewed as a direct consequence of Theorem 3.1 in [16] without proofs. A self-contained and complete proof is given in Appendix B of this paper.

3.2 Iterative design for lower-triangular systems

To achieve the global robust stabilization for the system (15), we first introduce the following assumption.

Assumption 2 For system (15), the following two conditions hold.

1) For $1 \leqslant i \leqslant n$, there are iISS Lyapunov functions $V_i^a := V_i^a(t, z_i)$ and $V_i^b := V_i^b(t, \zeta_i)$ satisfying, along trajectoriesof (15),

[2]Note that, we omit the trivial case $\gamma_{l,k} = 0$ in (19b), (19a), (19c) for the sake of simplicity. In addition, these nonzero functions may rely on $\alpha_1, \cdots, \alpha_{\min\{k,l\}}$. Moreover, throughout this paper, we use and refer the same mathematical notation and definitions as those in [16] and their explanations are omitted.

$$
\begin{cases}
\underline{\alpha}_i^a(\|z_i\|) \leqslant V_i^a(t,z_i) \leqslant \bar{\alpha}_i^a(\|z_i\|),\\
\dot{V}_i^a \leqslant \sum\limits_{j=1}^{i}[\gamma_{i,j}^a(V_j^a)+\gamma_{i,j}^b(V_j^b)+\gamma_{i,j}^c(\|x_j\|^2)],\\
\underline{\alpha}_i^b(\|\zeta_i\|) \leqslant V_i^b(t,\zeta_i) \leqslant \bar{\alpha}_i^b(\|\zeta_i\|),\\
\dot{V}_i^b \leqslant \sum\limits_{j=1}^{i}[\delta_{i,j}^a(V_j^a)+\delta_{i,j}^b(V_j^b)+\delta_{i,j}^c(\|x_j\|^2)],
\end{cases} \tag{21}
$$

where

$$
\begin{aligned}
&\gamma_{i,i}^a(V_i^a) = -\alpha_i^a(V_i^a),\ \delta_{i,i}^b(V_i^b) = -\alpha_i^b(V_i^b),\\
&\alpha_i^a \in \mathcal{K}_\infty,\ \ \alpha_i^b \in \mathcal{K}^o,\ \gamma_{i,j}^a, \delta_{i,j}^a \in \mathcal{K}\cap\mathcal{O}(\alpha_j),\\
&\gamma_{i,i}^b \equiv 0,\ \gamma_{i,j}^b, \delta_{i,j}^b \in \mathcal{K}^o\cap\mathcal{O}(\alpha_j),\\
&\gamma_{i,j}^c,\ \delta_{i,j}^c \in \mathcal{K}\cap\mathcal{O}(Id),
\end{aligned}
$$

2) For $1 \leqslant i \leqslant n$, the function $f_i^c(z_{[i]},\zeta_{[i]},x_{[i]},\mu)$ satisfies

$$
\begin{aligned}
&\|f_i^c(z_{[i]},\zeta_{[i]},x_{[i]},\mu)\|^2 \leqslant\\
&\sum_{j=1}^{i}[\psi_{i,j}^a(V_j^a)+\psi_{i,j}^b(V_j^b)+\psi_{i,j}^c(\|x_j\|^2)],
\end{aligned} \tag{22}
$$

for some $\psi_{i,j}^a \in \mathcal{K}\cap\mathcal{O}(\alpha_j)$, $\psi_{i,j}^b \in \mathcal{K}^o\cap\mathcal{O}(\alpha_j)$ and $\psi_{i,j}^c \in \mathcal{K}\cap\mathcal{O}(Id)$ for $1 \leqslant j \leqslant i$.

Proposition 1 Consider system (15) under Assumption 2. Then, there is a smooth controller of the form (16) such that the closed-loop system is globally robustly asymptotically stable at the origin.

The result of Proposition 1 is an immediate consequence from Theorem 3.2 in [16]. A self-contained and complete proof of Proposition 1 is given in Appendix C of this paper.

4 Simulation setup and results

In this section, let us continue to illustrate the proposed the stabilization method with the car-following system example elaborated in Section 2.

For numerical tests, we consider a string of $N = 5$ cars and a virtual commanding source as the leader vehicle. The nominal values of the vehicles' parameters are set as $m_i = 130$ kg, $A_{\rho i} = 0.3$ Ns^2/m^2, $d_i = 10$ N, $\tau_i = 0.2$ s, for $i = 1,2,\cdots,5$. The motion of the virtual lead vehicle is $p_0(t) = 150 + 30t + 30\sin(\frac{\pi}{30}t)$. The desired inner vehicle distance is set as $L_i = 30$ m, for $i = 1,\cdots,5$. The initial states of the following cars are set as $(p_i(0), v_i(0), f_i(0)) = (149 - \sum_{j=1}^{i} L_i + i, 40+i, 0)$, $i = 1,\cdots,5$.

The procedure of designing internal model-based longitudinal controllers for car-following systems (1) is summarized by Algorithm 1. The first step of designing internal models for steady-state compensation was presented in Section 2. In this simulation, we set the internal model (8) with $m_{i,1} = [1\ \ 2.15\ \ 1.75]^{\mathrm{T}}$, $m_{i,2} = m_{i,3} = [1\ \ 3.4\ \ 5.5\ \ 5\ \ 2.8]^{\mathrm{T}}$ for $i = 1,\cdots,5$.

To achieve the problem conversion, we write the translated augmented system (13) as (15) with

$$
\begin{aligned}
z_i &= \mathrm{col}(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,3}^a),\\
\zeta_i &= \mathrm{col}(\tilde{\eta}_{i,1}^b, \tilde{\eta}_{i,2}^b, \tilde{\eta}_{i,3}^b),\\
x_i &= \mathrm{col}(e_i, \bar{x}_{i,2}, \bar{x}_3, \bar{x}_4),
\end{aligned}
$$

Then the stabilizer in the 2nd step is designed as

$$x_{i,4} = -50(10 + 10\tilde{x}_{i,3}^2)\tilde{x}_{i,3},$$

with

$$
\begin{aligned}
\tilde{x}_{i,3} &= x_{i,3} + 50(10 + 10\tilde{x}_{i,2}^2)\tilde{x}_{i,2},\\
\tilde{x}_{i,2} &= x_{i,2} - 10x_{i,1},
\end{aligned}
$$

for $i = 1,\cdots,5$. It can be see from Figure 2 that the tracking errors e_i of all the following cars tend to zero asymptotically, which confirm our results in Proposition 1.

Algorithm 1 Internal model principle-based longitudinal controller for car-following systems (1)

for $1 \leqslant i \leqslant N$ **do**
  procedure COMPENSATION $x_{i,j+1}$
    Solve the regulator equations (7)
    for $1 \leqslant j \leqslant 3$ **do**
      Construct the internal model (8)
      Compute the internal model output by (10)
      return the internal model output by (10)
    end for
  end procedure
  Problem conversion
  procedure STABILIZATION $\bar{x}_{i,j}$
    Verify the conditions in Assumption 2
    Design the stabilizer for (15) $\triangleright$ Proposition 1
  end procedure
end for

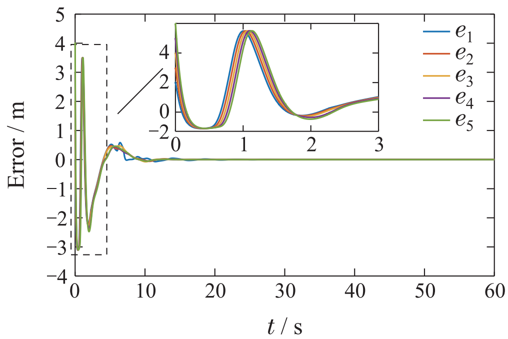

Fig. 2 Spacing errors e_i for the 5 following cars

5 Conclusion

We have presented a sufficient condition for stabilizing control of a class of networked nonlinear systems with dynamic uncertainties. The study has been motivated by and the proposed results have been applied to solve a longitudinal control problem for a string of automated cars moving in a lane. We have shown some simulation results to illustrate the proposed results.

The impact of the present study is to provide interesting stabilization design techniques for resolving more general control problems. Such problems arise in large-scale and multi-agent systems control for achieving the celebrated control goals such as consensus, synchronization, and formation in distributed networked control settings. Another future direction is to further apply the learning internal model-based method of [25] together with the proposed stabilization method to revisit the longitudinal platooning control problem.

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Appendix A Description of the translated augmented system

A detailed calculation for deriving the translated augmented system is as follows.

The time derivative of e_i satisfies

$$\begin{aligned}\dot{e}_i = {} & x_{i,2} - x^*_{i,2} = \\ & \bar{x}_{i,2} + \Gamma_{i,1}(\eta^a_{i,1}, \eta^b_{i,1}) - \Gamma_{i,1}(\theta^a_{i,1}, \theta^b_{i,1}) =: \\ & \bar{x}_{i,2} + \tilde{\Gamma}_{i,1}(\tilde{\eta}^a_{i,1}, \tilde{\eta}^b_{i,1}, e_i, \mu),\end{aligned}$$

where

$$\tilde{\Gamma}_{i,1} = \Gamma_{i,1}(\tilde{\eta}^a_{i,1} + \theta^a_{i,1} + N_{i,1}e_i, \tilde{\eta}^b_{i,1} + \theta^b_{i,1}) - \Gamma_{i,1}(\theta^a_{i,1}, \theta^b_{i,1}).$$

The time derivative of $\tilde{\eta}^a_{i,1}$ satisfies

$$\begin{aligned}\dot{\tilde{\eta}}^a_{i,1} = {} & \gamma^a_{i,1}(\eta^a_{i,1}) + N_{i,1}x_{i,2} - \gamma^a_{i,1}(\theta^a_{i,1}) - N_{i,1}x^*_{i,2} - \\ & N_{i,1}(x_{i,2} - x^*_{i,2}) = \\ & \gamma^a_{i,1}(\eta^a_{i,1}) - \gamma^a_{i,1}(\theta^a_{i,1}) = \\ & \gamma^a_{i,1}(\tilde{\eta}^a_{i,1} + \theta^a_{i,1} + N_{i,1}e_i) - \gamma^a_{i,1}(\theta^a_{i,1}) = \\ & M_{i,1}\tilde{\eta}^a_{i,1} + M_{i,1}N_{i,1}e_i.\end{aligned}$$

The time derivative of $\tilde{\eta}^a_{i,1}$ satisfies

$$\begin{aligned}\dot{\tilde{\eta}}^b_{i,1} = {} & \gamma^b_{i,1}(\eta^a_{i,1}, \eta^b_{i,1}) - \gamma^b_{i,1}(\theta^a_{i,1}, \theta^b_{i,1}) = \\ & \gamma^b_{i,1}(\tilde{\eta}^a_{i,1} + \theta^a_{i,1} + N_{i,1}e_i, \tilde{\eta}^b_{i,1} + \theta^b_{i,1}) - \gamma^b_{i,1}(\theta^a_{i,1}, \theta^b_{i,1}) = \\ & -(\tilde{\eta}^{a1}_{i,1} + \theta^{a1}_{i,1})[(\tilde{\eta}^{a1}_{i,1} + \theta^{a1}_{i,1})^{\mathrm{T}}(\tilde{\eta}^b_{i,1} + \theta^b_{i,1}) - \\ & (\tilde{\eta}^{a2}_{i,1} + \theta^{a2}_{i,1} + e_i)] + \theta^{a1}_{i,1}[(\theta^{a1}_{i,1})^{\mathrm{T}}\theta^b_{i,1} - \theta^{a2}_{i,1}] =: \\ & -\Theta_{i,1}(\mu)\tilde{\eta}^b_{i,1} + \varphi^b_{i,1}(\theta^a_{i,1}, \theta^b_{i,1}, e_i, \mu),\end{aligned}$$

where

$$\Theta_{i,1} = \theta^{a1}_{i,1}(\theta^{a1}_{i,1})^{\mathrm{T}}$$

$$\begin{aligned}\varphi_{i,1}^b = {} & \theta_{i,1}^{a1}[(\theta_{i,1}^{a1})^{\mathrm{T}}\theta_{i,1}^b - \theta_{i,1}^{a2}] - (\tilde{\eta}_{i,1}^{a1} + \theta_{i,1}^{a1})(\tilde{\eta}_{i,1}^{a1} + \theta_{i,1}^{a1})^{\mathrm{T}}\theta_{i,1}^b - \\ & [\tilde{\eta}_{i,1}^{a1}(\tilde{\eta}_{i,1}^{a1})^{\mathrm{T}} + \tilde{\eta}_{i,1}^{a1}(\theta_{i,1}^{a1})^{\mathrm{T}} + \theta_{i,1}^{a1}(\tilde{\eta}_{i,1}^{a1})^{\mathrm{T}}]\theta_{i,1}^b + \\ & (\tilde{\eta}_{i,1}^{a1} + \theta_{i,1}^{a1})(\tilde{\eta}_{i,1}^{a2} + \theta_{i,1}^{a2} + e_i).\end{aligned}$$

By using $x_{i,2}^* = \Gamma_{i,1}(\theta_{i,1}^a, \theta_{i,1}^b)$, $x_{i,3}^* = \Gamma_{i,2}(\theta_{i,2}^a, \theta_{i,2}^b)$, and $x_{i,3}^* = m_i \dfrac{\mathrm{d}x_{i,2}^*}{\mathrm{d}t} + A_{\rho i}x_{i,2}^{*2} + d_i$, the time derivative of $\bar{x}_{i,2}$ satisfies

$$\begin{aligned}\dot{\bar{x}}_{i,2} = {} & \frac{1}{m_i}x_{i,3} + \frac{1}{m_i}(-A_{\rho i}x_{i,2}^2 - d_i) - \frac{\mathrm{d}\Gamma_{i,1}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t} = \\ & \frac{1}{m_i}(\bar{x}_{i,3} + \Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)) + \frac{1}{m_i}(-A_{\rho i}x_{i,2}^2 - d_i) - \\ & \frac{\mathrm{d}\Gamma_{i,1}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t} = \\ & \frac{1}{m_i}\bar{x}_{i,3} + \frac{1}{m_i}(\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b) - \Gamma_{i,2}(\theta_{i,2}^a, \theta_{i,2}^b)) - \\ & \frac{\mathrm{d}\Gamma_{i,1}(\eta_{i,1}^a, \eta_{i,1}^b)}{\mathrm{d}t} + \frac{\mathrm{d}\Gamma_{i,1}(\theta_{i,1}^a, \theta_{i,1}^b)}{\mathrm{d}t} + \\ & \frac{1}{m_i}(A_{\rho i}x_{i,2}^{*2} - A_{\rho i}x_{i,2}^2) =: \\ & \frac{1}{m_i}\bar{x}_{i,3} + \frac{1}{m_i}\tilde{\Gamma}_{i,2}(\tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, \bar{x}_{i,2}, \mu) - \\ & \tilde{\Gamma}_{i,1}'(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \bar{x}_{i,2}, \mu) + \rho_{i,2}(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \bar{x}_{i,2}, \mu),\end{aligned}$$

where

$$\begin{aligned}\tilde{\Gamma}_{i,2} = {} & \Gamma_{i,2}(\tilde{\eta}_{i,2}^a + \theta_{i,2}^a + m_iN_{i,2}\bar{x}_{i,2}, \tilde{\eta}_{i,2}^b + \theta_{i,2}^b) - \\ & \Gamma_{i,2}(\theta_{i,2}^a, \theta_{i,2}^b), \\ \tilde{\Gamma}_{i,1}' = {} & \frac{\partial\Gamma_{i,1}(\eta_{i,1}^a, \eta_{i,1}^b)}{\partial\eta_{i,1}^a}(\dot{\tilde{\eta}}_{i,1}^a + \dot{\theta}_{i,1}^a + N_{i,1}\dot{e}_i) + \\ & \frac{\partial\Gamma_{i,1}(\eta_{i,1}^a, \eta_{i,1}^b)}{\partial\eta_{i,1}^b}(\dot{\tilde{\eta}}_{i,1}^b + \dot{\theta}_{i,1}^b) - \frac{\mathrm{d}\Gamma_{i,1}(\theta_{i,1}^a, \theta_{i,1}^b)}{\mathrm{d}t}, \\ \rho_{i,2} = {} & \frac{1}{m_i}A_{\rho i}(\Gamma_{i,1}(\theta_{i,1}^a, \theta_{i,1}^b))^2 - \frac{1}{m_i}A_{\rho i}(\bar{x}_{i,2} + \\ & \Gamma_{i,1}(\tilde{\eta}_{i,1}^a + \theta_{i,1}^a + N_{i,1}e_i, \tilde{\eta}_{i,1}^b + \theta_{i,1}^b))^2.\end{aligned}$$

The time derivative of $\tilde{\eta}_{i,2}^a$ satisfies

$$\begin{aligned}\dot{\tilde{\eta}}_{i,2}^a = {} & \gamma_{i,2}^a(\eta_{i,2}^a) + N_{i,2}x_{i,3} - \gamma_{i,2}^a(\theta_{i,2}^a) - N_{i,2}x_{i,3}^* - \\ & m_iN_{i,2}(\dot{x}_{i,2} - \frac{\mathrm{d}\Gamma_{i,1}(\eta_{i,1}^a, \eta_{i,1}^b)}{\mathrm{d}t}) = \\ & \gamma_{i,2}^a(\eta_{i,2}^a) + N_{i,2}x_{i,3} - \gamma_{i,2}^a(\theta_{i,2}^a) - N_{i,2}x_{i,3}^* - \\ & N_{i,2}(x_{i,3} - A_{\rho i}x_{i,2}^2 - d_i - m_i\frac{\mathrm{d}\Gamma_{i,1}(\eta_{i,1}^a, \eta_{i,1}^b)}{\mathrm{d}t}) = \\ & \gamma_{i,2}^a(\eta_{i,2}^a) - \gamma_{i,2}^a(\theta_{i,2}^a) - N_{i,2}(-A_{\rho i}x_{i,2}^2 + A_{\rho i}x_{i,2}^{*2} + \\ & m_i\frac{\mathrm{d}x_{i,2}^*}{\mathrm{d}t} - m_i\frac{\mathrm{d}\Gamma_{i,1}(\eta_{i,1}^a, \eta_{i,1}^b)}{\mathrm{d}t}) =: \\ & M_{i,2}\tilde{\eta}_{i,2}^a + m_iM_{i,2}N_{i,2}\bar{x}_{i,2} + \\ & N_{i,2}\rho_{i,2}(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \bar{x}_{i,2}, \mu) - \\ & N_{i,2}\tilde{\Gamma}_{i,1}'(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, e_i, \bar{x}_{i,2}, \mu).\end{aligned}$$

The time derivative of $\tilde{\eta}_{i,2}^b$ satisfies

$$\begin{aligned}\dot{\tilde{\eta}}_{i,2}^b = {} & \gamma_{i,2}^b(\eta_{i,2}^a, \eta_{i,2}^b) - \gamma_{i,2}^b(\theta_{i,2}^a, \theta_{i,2}^b) = \\ & \gamma_{i,2}^b(\tilde{\eta}_{i,2}^a + \theta_{i,2}^a + m_iN_{i,2}\bar{x}_{i,2}, \tilde{\eta}_{i,2}^b + \theta_{i,2}^b) - \\ & \gamma_{i,2}^b(\theta_{i,2}^a, \theta_{i,2}^b) =: \\ & -\Theta_{i,2}(\mu)\tilde{\eta}_{i,2}^b + \varphi_{i,2}^b(\theta_{i,2}^a, \theta_{i,2}^b, \bar{x}_{i,2}, \mu),\end{aligned}$$

where

$$\begin{aligned}\Theta_{i,2} = {} & \theta_{i,2}^{a1}(\theta_{i,2}^{a1})^{\mathrm{T}}, \\ \varphi_{i,2}^b = {} & \theta_{i,2}^{a1}[(\theta_{i,2}^{a1})^{\mathrm{T}}\theta_{i,2}^b - \theta_{i,2}^{a2}] - \\ & (\tilde{\eta}_{i,2}^{a1} + \theta_{i,2}^{a1})(\tilde{\eta}_{i,2}^{a1} + \theta_{i,2}^{a1})^{\mathrm{T}}\theta_{i,2}^b - \\ & [\tilde{\eta}_{i,2}^{a1}(\tilde{\eta}_{i,2}^{a1})^{\mathrm{T}} + \tilde{\eta}_{i,2}^{a1}(\theta_{i,2}^{a1})^{\mathrm{T}} + \theta_{i,2}^{a1}(\tilde{\eta}_{i,2}^{a1})^{\mathrm{T}}]\theta_{i,2}^b + \\ & (\tilde{\eta}_{i,2}^{a1} + \theta_{i,2}^{a1})(\tilde{\eta}_{i,2}^{a2} + \theta_{i,2}^{a2} + m_ix_{i,2}).\end{aligned}$$

By using $x_{i,4}^* = \Gamma_{i,3}(\theta_{i,3}^a, \theta_{i,3}^b)$, $x_{i,3}^* = \Gamma_{i,2}(\theta_{i,2}^a, \theta_{i,2}^b)$ and $x_{i,4}^* = \tau_i\dfrac{\mathrm{d}x_{i,3}^*}{\mathrm{d}t} + x_{i,3}^*$, the time derivative of $\bar{x}_{i,3}$ satisfies

$$\begin{aligned}\dot{\bar{x}}_{i,3} = {} & \frac{1}{\tau_i}x_{i,4} - \frac{1}{\tau_i}x_{i,3} - \frac{\mathrm{d}\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t} = \\ & \frac{1}{\tau_i}(\bar{x}_{i,4} + \Gamma_{i,3}(\eta_{i,3}^a, \eta_{i,3}^b)) - \\ & \frac{1}{\tau_i}x_{i,3} - \frac{\mathrm{d}\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t} = \\ & \frac{1}{\tau_i}\bar{x}_{i,4} + \frac{1}{\tau_i}(\Gamma_{i,3}(\eta_{i,3}^a, \eta_{i,3}^b) - \Gamma_{i,3}(\theta_{i,3}^a, \theta_{i,3}^b)) - \\ & \frac{\mathrm{d}\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t} + \frac{\mathrm{d}\Gamma_{i,2}(\theta_{i,2}^a, \theta_{i,2}^b)}{\mathrm{d}t} + \\ & \frac{1}{\tau_i}(x_{i,3}^* - x_{i,3}) := \\ & \frac{1}{\tau_i}\bar{x}_{i,4} + \frac{1}{\tau_i}\tilde{\Gamma}_{i,3}(\tilde{\eta}_{i,3}^a, \tilde{\eta}_{i,3}^b, \bar{x}_{i,3}, \mu) - \\ & \tilde{\Gamma}_{i,2}'(\tilde{\eta}_{i,1}^a, \tilde{\eta}_{i,1}^b, \tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, e_i, \bar{x}_{i,2}, \mu) + \\ & \rho_{i,3}(\tilde{\eta}_{i,2}^a, \tilde{\eta}_{i,2}^b, \bar{x}_{i,2}, \bar{x}_{i,3}, \mu),\end{aligned}$$

where

$$\begin{aligned}\tilde{\Gamma}_{i,3} = {} & \Gamma_{i,3}(\tilde{\eta}_{i,3}^a + \theta_{i,3}^a + \tau_iN_{i,3}\bar{x}_{i,3}, \tilde{\eta}_{i,3}^b + \theta_{i,3}^b) - \\ & \Gamma_{i,3}(\theta_{i,3}^a, \theta_{i,3}^b), \\ \tilde{\Gamma}_{i,2}' = {} & \frac{\partial\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\partial\eta_{i,2}^a}(\dot{\tilde{\eta}}_{i,2}^a + \dot{\theta}_{i,2}^a + m_iN_{i,2}\dot{\bar{x}}_{i,2}) + \\ & \frac{\partial\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\partial\eta_{i,2}^b}(\dot{\tilde{\eta}}_{i,2}^b + \dot{\theta}_{i,2}^b) - \frac{\mathrm{d}\Gamma_{i,2}(\theta_{i,2}^a, \theta_{i,2}^b)}{\mathrm{d}t}, \\ \rho_{i,3} = {} & \frac{1}{\tau_i}(x_{i,3}^* - \bar{x}_{i,3} - \\ & \Gamma_{i,2}(\tilde{\eta}_{i,2}^a + \theta_{i,2}^a + m_iN_{i,2}\bar{x}_{i,2}, \tilde{\eta}_{i,2}^b + \theta_{i,2}^b)).\end{aligned}$$

The time derivative of $\tilde{\eta}_{i,3}^a$ satisfies

$$\begin{aligned}\dot{\eta}_{i,3}^a = {} & \gamma_{i,3}^a(\eta_{i,3}^a) + N_{i,3}x_{i,4} - \gamma_{i,3}^a(\theta_{i,3}^a) - N_{i,3}x_{i,4}^* - \\ & \tau_iN_{i,3}(\dot{x}_{i,3} - \frac{\mathrm{d}\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t}) = \\ & \gamma_{i,3}^a(\eta_{i,3}^a) + N_{i,3}x_{i,4} - \gamma_{i,3}^a(\theta_{i,3}^a) - N_{i,3}x_{i,4}^* - \\ & N_{i,3}(x_{i,4} - x_{i,3} - \tau_i\frac{\mathrm{d}\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t}) = \\ & \gamma_{i,3}^a(\eta_{i,3}^a) - \gamma_{i,3}^a(\theta_{i,3}^a) - N_{i,3}(\tau_i\frac{\mathrm{d}x_{i,3}^*}{\mathrm{d}t} + x_{i,3}^* - \\ & x_{i,3} - \tau_i\frac{\mathrm{d}\Gamma_{i,2}(\eta_{i,2}^a, \eta_{i,2}^b)}{\mathrm{d}t}) := \\ & M_{i,3}\tilde{\eta}_{i,3}^a + \tau_iM_{i,3}N_{i,3}x_{i,3} +\end{aligned}$$

$$N_{i,3}\tilde{\Gamma}'_{i,2}(\tilde{\eta}^a_{i,1},\tilde{\eta}^b_{i,1},\tilde{\eta}^a_{i,2},\tilde{\eta}^b_{i,2},e_i,\bar{x}_{i,2},\mu)-N_{i,3}\rho_{i,3}(\tilde{\eta}^a_{i,2},\tilde{\eta}^b_{i,2},\bar{x}_{i,2},\bar{x}_{i,3},\mu).$$

The time derivative of $\tilde{\eta}^b_{i,3}$ satisfies

$$\begin{aligned}\dot{\eta}^b_{i,3}=&\ \gamma^b_{i,3}(\eta^a_{i,3},\eta^b_{i,3})-\gamma^b_{i,3}(\theta^a_{i,3},\theta^b_{i,3})=\\ &\gamma^b_{i,3}(\tilde{\eta}^a_{i,3}+\theta^a_{i,3}+\tau_i N_{i,3}\bar{x}_{i,3},\tilde{\eta}^b_{i,3}+\theta^b_{i,3})-\\ &\gamma^b_{i,3}(\theta^a_{i,3},\theta^b_{i,3}):=\\ &-\Theta_{i,3}(\mu)\tilde{\eta}^b_{i,3}+\varphi^b_{i,3}(\theta^a_{i,3},\theta^b_{i,3},\bar{x}_{i,3},\mu),\end{aligned}$$

where

$$\begin{aligned}\Theta_{i,3}=&\ \theta^{a1}_{i,3}(\theta^{a1}_{i,3})^{\mathrm{T}},\\ \varphi^b_{i,3}=&\ \theta^{a1}_{i,3}[(\theta^{a1}_{i,3})^{\mathrm{T}}\theta^b_{i,3}-\theta^{a2}_{i,3}]-\\ &(\tilde{\eta}^{a1}_{i,3}+\theta^{a1}_{i,3})(\tilde{\eta}^{a1}_{i,3}+\theta^{a1}_{i,3})^{\mathrm{T}}\theta^b_{i,3}-\\ &[\tilde{\eta}^{a1}_{i,3}(\tilde{\eta}^{a1}_{i,3})^{\mathrm{T}}+\tilde{\eta}^{a1}_{i,3}(\theta^{a1}_{i,3})^{\mathrm{T}}+\theta^{a1}_{i,3}(\tilde{\eta}^{a1}_{i,3})^{\mathrm{T}}]\theta^b_{i,3}+\\ &(\tilde{\eta}^{a1}_{i,3}+\theta^{a1}_{i,3})(\tilde{\eta}^{a2}_{i,3}+\theta^{a2}_{i,3}+\tau_i x_{i,3}).\end{aligned}$$

Appendix B Proof of Lemma 1

The proof can be done by Mathematical Induction of verifying conditions in [12, Theorem 3]. Two lemmas are given below respectively. In the sequel, Lemma 2 verifies the initial step $i=1$ and Lemma 3 demonstrates the induction from i to $i+1$ for all $1\leqslant i\leqslant n-1$ in the case $n>1$. For the sake of convenience, for $1\leqslant i\leqslant n$, denote that

$$\begin{cases}\boldsymbol{A}^{[i]}(\nu):=[\alpha_1(\nu_1)\ \cdots\ \alpha_{3i}(\nu_{3i})]^{\mathrm{T}},\\ \digamma^{[i]}(\nu):=[\sum\limits_{k=1,k\neq1}^{3i}\gamma_{1,k}(\nu_k)\ \cdots\ \sum\limits_{k=1,k\neq3i}^{3i}\gamma_{3i,k}(\nu_k)]^{\mathrm{T}}\end{cases}\tag{B1}$$

and

$$\begin{cases}\boldsymbol{D}^{[i]}(\nu):=\epsilon_i^{-1}[\omega^*_1\nu_1\ \cdots\ \omega^*_{3i}\nu_{3i}]^{\mathrm{T}},\\ \Lambda^{[i]}(\nu):=[\lambda^{[i]}_1(\nu_1)\ \cdots\ \lambda^{[i]}_{3i}(\nu_{3i})]^{\mathrm{T}},\end{cases}\tag{B2}$$

where

$$0<\epsilon_i<1,\ \omega^*_k>1,\ \lambda^{[i]}_k(\nu_k)\in\mathcal{N},\ 1\leqslant k\leqslant 3i.$$

Lemma 2 At the step $i=1$, consider the network (17) with $m=3$. Under Assumption 1, for any

$$0<\epsilon_1<1,\ \omega^*_k>1,\ 1\leqslant k\leqslant 3,$$

there are

$$\alpha_3\in\mathcal{K}\cap\mathcal{O}(Id),\ \Lambda^{[1]}(\nu):=[\lambda_1(\nu_1)\ \ \lambda^*_2\ \ 1]$$

with $\lambda_1\in\mathcal{N}$ and $\lambda^*_2>0$, satisfying the following condition:

$$\Lambda^{[1]}(\nu)\digamma^{[1]}(\nu)\leqslant\Lambda^{[1]}(\nu){\boldsymbol{D}^{[1]}}^{-1}\circ\boldsymbol{A}^{[1]}(\nu),\ \forall\nu\in\mathbb{R}^3_+,\tag{B3}$$

Proof First note that, under Assumption 1,

$$\begin{aligned}&\Lambda^{[1]}(\nu)\digamma^{[1]}(\nu)=\\ &\lambda_1(\nu_1)\gamma_{1,3}(\nu_3)+\lambda^*_2[\gamma_{2,1}(\nu_1)+\gamma_{2,3}(\nu_3)]+\\ &\gamma_{3,1}(\nu_1)+\gamma_{3,2}(\nu_2),\end{aligned}$$

$$\begin{aligned}&\Lambda^{[1]}(\nu){\boldsymbol{D}^{[1]}}^{-1}\circ\boldsymbol{A}^{[1]}(\nu)=\\ &\epsilon_1[\lambda_1(\nu_1)\omega_1^{*-1}\alpha_1(\nu_1)+\lambda^*_2\omega_2^{*-1}\alpha_2(\nu_2)+\\ &\omega_3^{*-1}\alpha_3(\nu_3)].\end{aligned}$$

In the above, pick $\lambda_1(s):=\lambda^*_1+\lambda'_1(s)$ for $s\geqslant 0$ with

$$\lambda^*_1:=\lambda_1(0)>0,\ \lambda'_1\in\mathcal{K}_\infty.\tag{B4}$$

Then by Young's inequality, it follows, for any $\psi_1\in\mathcal{K}_\infty$

$$\begin{aligned}\lambda_1(\nu_1)\gamma_{1,3}(\nu_3)=&\ \lambda^*_1\gamma_{1,3}(\nu_3)+\lambda'_1(\nu_1)\gamma_{1,3}(\nu_3)\leqslant\\ &\lambda^*_1\gamma_{1,3}(\nu_3)+\psi_1\circ\lambda'_1(\nu_1)\cdot\lambda'_1(\nu_1)+\\ &\psi_1^{-1}\circ\gamma_{1,3}(\nu_3)\cdot\gamma_{1,3}(\nu_3).\end{aligned}$$

Thus, to show (B3), it suffices to find $\alpha_3\in\mathcal{K}_\infty$, $\lambda^*_1,\lambda^*_2>0$ and $\lambda'_1\in\mathcal{K}_\infty$ such that for all $\nu\in\mathbb{R}^3_+$

$$\begin{aligned}&\psi_1\circ\lambda'_1(\nu_1)\cdot\lambda'_1(\nu_1)+\lambda^*_2\gamma_{2,1}(\nu_1)+\gamma_{3,1}(\nu_1)\leqslant\\ &\epsilon_1\omega_1^{*-1}\lambda_1(\nu_1)\alpha_1(\nu_1),\end{aligned}\tag{B5a}$$

$$\gamma_{3,2}(\nu_2)\leqslant\epsilon_1\omega_2^{*-1}\lambda^*_2\alpha_2(\nu_2),\tag{B5b}$$

$$\begin{aligned}&\lambda^*_1\gamma_{1,3}(\nu_3)+\lambda^*_2\gamma_{2,3}(\nu_3)+\psi_1^{-1}\circ\gamma_{1,3}(\nu_3)\cdot\gamma_{1,3}(\nu_3)\leqslant\\ &\epsilon_1\omega_3^{*-1}\alpha_3(\nu_3).\end{aligned}\tag{B5c}$$

To do so, notice that existence of α_3 in (B5c) is straightforward as long as its left-hand side functions are determined. The proofs of (B5a) and (B5b) are given below.

Proof of (B5b). By using [17, Lemma 3.1], for $\gamma_{3,2}\in(\mathcal{K}^o\cup\{0\})\cap\mathcal{O}(\alpha_2)$ with $\alpha_2\in\mathcal{K}^o$, there exists a constant $\lambda^*_2>0$ satisfying (B5b).

Proof of (B5a). Let

$$\psi_1(s)=\frac{1}{2}\epsilon_1\omega_1^{*-1}\alpha_1\circ\lambda_1'^{-1}(s),$$

which, together with $\lambda'_1(s)\leqslant\lambda_1(s)$ for $s\geqslant 0$, gives

$$\psi_1\circ\lambda'_1(\nu_1)\cdot\lambda'_1(\nu_1)\leqslant\frac{1}{2}\epsilon_1\omega_1^{*-1}\lambda_1(\nu_1)\alpha_1(\nu_1).$$

Since $\gamma_{2,1},\gamma_{3,1}\in(\mathcal{K}\cup\{0\})\cap\mathcal{O}(\alpha_1)$, by using [26, Lemma 1], there exists a function $\lambda_1\in\mathcal{N}$ such that

$$\lambda^*_2\gamma_{2,1}(\nu_1)+\gamma_{3,1}(\nu_1)\leqslant\frac{1}{2}\epsilon_1\omega_1^{*-1}\lambda_1(\nu_1)\alpha_1(\nu_1),$$

which confirms (B5a). □

To present the following induction lemma from $i-1$ to i, we introduce the following induction hypothesis.

Induction Hypothesis. At the step i for $1\leqslant i\leqslant n-1$, consider the network (17) with $m=3i$. For any

$$0<\epsilon_1<\cdots<\epsilon_i<1,\ \omega^*_k>1,\ 1\leqslant k\leqslant 3i,$$

there exist

$$\begin{aligned}&\alpha_3,\cdots,\alpha_{3(i-1)}\in\mathcal{K}\cap\mathcal{O}(Id),\\ &\Lambda^{[i]}(\nu)=[\lambda^{[i]}_1(\nu)\ \cdots\ \lambda^{[i]}_{3i}(\nu)],\end{aligned}$$

satisfy the condition

$$\Lambda^{[i]}(\nu)\digamma^{[i]}(\nu)\leqslant\Lambda^{[i-1]}(\nu){\boldsymbol{D}^{[i]}}^{-1}\circ\boldsymbol{A}^{[i]}(\nu),\ \forall\nu\in\mathbb{R}^{3i}_+,\tag{B6}$$

Lemma 3 At the step $i+1$ for $1 \leqslant i \leqslant n-1$, consider the network (17) with $m = 3(i+1)$. Under Assumption 1, suppose that, at step i, the ***Induction Hypothesis*** is ensured. Then, for any

$$0 < \epsilon_i < \epsilon_{i+1} < 1,\ \omega_k^* > 1,\ 3i+1 \leqslant k \leqslant 3i+3,$$

there are

$$\alpha_3, \cdots, \alpha_{3i}, \alpha_{3i+3} \in \mathcal{K} \cap \mathcal{O}(Id),$$
$$\Lambda^{[i+1]}(\nu) := [\delta^{[i]}(\nu)\Lambda^{[i]}(\nu)\ \ \lambda_{3i+1}(\nu_{3i+1})\ \ \lambda_{3i+2}^*\ \ 1],$$

with $\lambda_{3i+1} \in \mathcal{N}$, $\lambda_{3i+2}^* > 0$, and

$$\delta^{[i]}(\nu) = \sum_{k=1}^{3i} \delta_k^{[i]}(\nu_k),$$
$$\delta_k^{[i]} \begin{cases} \in \mathcal{N}, & \text{if } 1 \leqslant k \leqslant 3i,\ k \in \boldsymbol{n}_a \cup \boldsymbol{n}_c, \\ \equiv \delta_k^{[i]*}, & \text{if } 1 \leqslant k \leqslant 3i,\ k \in \boldsymbol{n}_b, \end{cases}$$

for the constant $\delta_k^{[i]*} > 0$, satisfying, for all $\nu \in \mathbb{R}_+^{3(i+1)}$,

$$\Lambda^{[i+1]}(\nu)\Gamma^{[i+1]}(\nu) \leqslant \Lambda^{[i+1]}(\nu)\boldsymbol{D}^{[i+1]^{-1}} \circ \boldsymbol{A}^{[i+1]}(\nu), \tag{B7}$$

Proof Recall the ***Induction Hypothesis*** for $m = 3i$ that has assured the existence of $\alpha_3, \cdots, \alpha_{3i} \in \mathcal{K} \cap \mathcal{O}(Id)$ satisfying condition (B6). In the following, existence of $\alpha_{3i+3} \in \mathcal{K} \cap \mathcal{O}(Id)$ is shown for the case $m = 3(i+1)$. Note that, under Assumption 1,

$$\begin{cases} \Lambda^{[i+1]}(\nu)\Gamma^{[i+1]}(\nu) = \\ \delta^{[i]}(\nu)\Lambda^{[i]}(\nu)\Gamma^{[i]}(\nu) + \delta^{[i]}(\nu)\Lambda^{[i]}(\nu)\gamma_{3i,3i+3}(\nu_{3i+3}) + \\ \lambda_{3i+1}(\nu_{3i+1}) \sum\limits_{k=1,k\neq 3i+1,3i+2}^{3i+3} \gamma_{3i+1,k}(\nu_k) + \\ \lambda_{3i+2}^* \sum\limits_{k=1,k\neq 3i+2}^{3i+3} \gamma_{3i+2,k}(\nu_k) + \sum\limits_{k=1,k\neq 3i+3}^{3i+3} \gamma_{3i+3,k}(\nu_k), \\ \Lambda^{[i+1]}(\nu)\boldsymbol{D}^{[i+1]^{-1}} \circ \boldsymbol{A}^{[i+1]}(\nu) = \\ \dfrac{\epsilon_{i+1}}{\epsilon_i}\delta^{[i]}(\nu)\Lambda^{[i]}(\nu)\boldsymbol{D}^{[i]^{-1}} \circ \boldsymbol{A}^{[i]}(\nu) + \\ \epsilon_{i+1}\lambda_{3i+1}(\nu_{3i+1})\omega_{3i+1}^{*-1}\alpha_{3i+1}(\nu_{3i+1}) + \\ \epsilon_{i+1}\lambda_{3i+2}^*\omega_{3i+2}^{*-1}\alpha_{3i+2}(\nu_{3i+2}) + \epsilon_{i+1}\omega_{3i+3}^{*-1}\alpha_{3i+3}(\nu_{3i+3}). \end{cases} \tag{B8}$$

Further pick $\delta_k^{[i]} \in \mathcal{N}$ for $1 \leqslant k \leqslant 3i$ and $k \in \boldsymbol{n}_a \cup \boldsymbol{n}_c$, and $\lambda_{3i+1} \in \mathcal{N}$ to be

$$\delta_k^{[i]}(s) := \delta_k^{[i]*} + \delta_k^{[i]'}(s),\ \delta_k^{[i]*} := \delta_k^{[i]}(0) > 0,$$
$$\lambda_{3i+1}(s) := \lambda_{3i+1}^* + \lambda_{3i+1}'(s),\ \forall s \geqslant 0,$$
$$\lambda_{3i+1}^* := \lambda_{3i+1}(0) > 0,\ \delta_k^{[i]'},\ \lambda_{3i+1}' \in \mathcal{K}_\infty.$$

Then by Young's Inequality, it gives, for $\psi_{1,1}^{[i]}, \psi_{1,3}^{[i]}, \psi_2^{[i]} \in \mathcal{K}_\infty$,

$$\begin{cases} \delta^{[i]}(\nu)\Lambda^{[i]}(\nu)\gamma_{3i,3i+3}(\nu_{3i+3}) \leqslant \\ \sum\limits_{k=1}^{3i} \delta_k^{[i]*}\gamma_{3i,3i+3}(\nu_{3i+3}) + \\ \sum\limits_{k=1,k\notin\boldsymbol{n}_b}^{3i} \psi_{1,k}^{[i]-1} \circ \delta_k^{[i]'}(\nu_k) \cdot \delta_k^{[i]'}(\nu_i) + \\ \sum\limits_{k=1,k\notin\boldsymbol{n}_b}^{3i} \psi_{1,k}^{[i]} \circ \gamma_{3i,3i+3}(\nu_{3i+3}) \cdot \gamma_{3i,3i+3}(\nu_{3i+3}), \\ \lambda_{3i+1}(\nu_{3i+1}) \sum\limits_{k=1,k\neq 3i+1,3i+2}^{3i+3} \gamma_{3i+1,k}(\nu_k) \leqslant \\ \lambda_{3i+1}^* \sum\limits_{k=1,k\neq 3i+1,3i+2}^{3i+3} \gamma_{3i+1,k}(\nu_k) + \\ (3i+1)\psi_2 \circ \lambda_{3i+1}'(\nu_{3i+1}) \cdot \lambda_{3i+1}'(\nu_{3i+1}) + \\ \sum\limits_{k=1,k\neq 3i+1,3i+2}^{3i+3} \psi_2^{-1} \circ \gamma_{3i+1,k}(\nu_k) \cdot \gamma_{3i+1,k}(\nu_k). \end{cases} \tag{B9}$$

Thus, in view of (B3), (B8) and (B9), to show (B7), it suffices to show that

$$\bar{\gamma}^{[i+1]}(\nu) \leqslant \frac{\epsilon_{i+1} - \epsilon_i}{\epsilon_i}\delta^{[i]}(\nu)\Lambda^{[i]}(\nu)\boldsymbol{D}^{[i]^{-1}} \circ \boldsymbol{A}^{[i]}(\nu), \tag{B10a}$$

$$\begin{aligned} &(3i+1)\psi_2 \circ \lambda_{3i+1}'(\nu_{3i+1}) \cdot \lambda_{3i+1}'(\nu_{3i+1}) + \\ &\lambda_{3i+2}^*\gamma_{3i+2,3i+1}(\nu_{3i+1}) + \gamma_{3i+3,3i+1}(\nu_{3i+1}) \leqslant \\ &\epsilon_{i+1}\lambda_{3i+1}(\nu_{3i+1})\omega_{3i+1}^{*-1}\alpha_{3i+1}(\nu_{43i+1}), \end{aligned} \tag{B10b}$$

$$\gamma_{3i+3,3i+2}(\nu_{3i+2}) \leqslant \epsilon_{i+1}\lambda_{3i+2}^*\omega_{3i+2}^{*-1}\alpha_{3i+2}(\nu_{3i+2}), \tag{B10c}$$

$$\gamma^{[i+1]}(\nu_{3i+3}) \leqslant \epsilon_{i+1}\omega_{3i+3}^{*-1}\alpha_{3i+3}(\nu_{3i+3}), \tag{B10d}$$

where

$$\begin{aligned} \bar{\gamma}^{[i+1]}(\nu) := &\sum_{k=1,k\notin\boldsymbol{n}_b}^{3i} \psi_{1,k}^{[i]-1} \circ \delta_k^{[i]'}(\nu_k) \cdot \delta_k^{[i]'}(\nu_k) + \\ &\sum_{k=1}^{3i} \psi_2^{[i]-1} \circ \gamma_{3i+1,k}(\nu_k) \cdot \gamma_{3i+1,k}(\nu_k) + \\ &\sum_{i=1}^{3i} [\lambda_{3i+1}^*\gamma_{3i+1,k}(\nu_k) + \\ &\lambda_{3i+2}^*\gamma_{3i+2,k}(\nu_k) + \gamma_{3i+3,k}(\nu_k)], \end{aligned}$$

$$\begin{aligned} \gamma^{[i+1]}(\nu_{3i+3}) := &\sum_{k=1}^{3i} \delta_k^{[i]*}\gamma_{3i,3i+3}(\nu_{3i+3}) + \\ &\sum_{k=1,k\notin\boldsymbol{n}_b}^{3i} \psi_{1,k}^{[i]} \circ \gamma_{3i,3i+3}(\nu_{3i+3}) \times \\ &\gamma_{3i,3i+3}(\nu_{3i+3}) + \\ &\lambda_{3i+1}^*\gamma_{3i+1,3i+3}(\nu_{3i+3}) + \\ &\lambda_{3i+2}^*\gamma_{3i+2,3i+3}(\nu_{3i+3}) + \\ &\psi_2^{[i]-1} \circ \gamma_{3i+1,3i+3}(\nu_{3i+3}) \times \\ &\gamma_{3i+1,3i+3}(\nu_{3i+3}). \end{aligned}$$

Existence of α_6 satisfying (B10d) is clear. The above (B10a) to (B10c) are shown below.

Proof of (B10c). By using [17, Lemma 3.1], for $\gamma_{3i+3,3i+2} \in \mathcal{K}^o \cap \mathcal{O}(\alpha_{3i+2})$ with $\alpha_{3i+2} \in \mathcal{K}^o$, there exists a constant $\lambda_{3i+2}^* > 0$ satisfying (B10c).

Proof of (B10b). Note that $\gamma_{3i+2,3i+1}, \gamma_{3i+3,3i+1} \in \mathcal{K}\cap\mathcal{O}(\alpha_{3i+1})$ and $\lambda'_{3i+2} \in \mathcal{K}_\infty$. By using [26, Lemma 1], there exists $\lambda_{3i+1} \in \mathcal{N}$ such that

$$\lambda^*_{3i+2}\gamma_{3i+2,3i+1}(\nu_{3i+1}) + \gamma_{3i+3,3i+1}(\nu_{3i+1}) \leqslant \frac{1}{2}\epsilon_{i+1}\lambda_{3i+1}(\nu_{3i+1})\omega^{*-1}_{3i+1}\alpha_{3i+1}(\nu_{3i+1}).$$

Further, let

$$\psi_2(s)^{[i]} := \frac{1}{2}\epsilon_{i+1}\omega^{*-1}_{3i+1}\alpha_{3i+1}\circ\lambda'^{-1}_{3i+1}(s) \in \mathcal{K}_\infty.$$

Then, it leads to

$$\begin{aligned}&(3i+1)\psi_2^{[i]}\circ\lambda'_{3i+1}(\nu_{3i+1})\cdot\lambda'_{3i+1}(\nu_{3i+1}) \leqslant \\ &\frac{1}{2}\epsilon_{i+1}\lambda'_{3i+1}(\nu_{3i+1})\omega^{*-1}_{3i+1}\alpha_{3i+1}(\nu_{3i+1}) \leqslant \\ &\frac{1}{2}\epsilon_{i+1}\lambda_{3i+1}(\nu_{3i+1})\omega^{*-1}_{3i+1}\alpha_{3i+1}(\nu_{3i+1}),\end{aligned}$$

which confirms (B10b).

Proof of (B10a). First, for $k \in \boldsymbol{n}_a \cup \boldsymbol{n}_c$, similar to the Proof of (B10b), since $\gamma_{l,k} \in \mathcal{K}\cap\mathcal{O}(\alpha_k)$, $3i+1 \leqslant l \leqslant 3i+3$, there exists $\delta_k \in \mathcal{N}$ such that

$$\begin{aligned}&\lambda^*_{3i+1}\gamma_{3i+1,k}(\nu_k) + \psi_2^{[i]-1}\circ\gamma_{3i+1,k}(\nu_i)\cdot\gamma_{3i+1,k}(\nu_k) + \\ &\lambda^*_{3i+2}\gamma_{3i+2,k}(\nu_k) + \gamma_{3i+3,k}(\nu_k) \leqslant \\ &\frac{1}{2}(\epsilon_{i+1}-\epsilon_i)\delta_k^{[i]}(\nu_k)\lambda_k^{[i]}(\nu_i)\omega_k^{*-1}\alpha_k(\nu_k).\end{aligned}$$

Second, let

$$\psi_{1,k}^{[i]}(s) := \frac{1}{2}(\epsilon_{i+1}-\epsilon_i)\omega_k^{*-1}(\lambda_k^{[i]}\cdot\alpha_k)\circ\delta_k^{[i]'-1}(s) \in \mathcal{K}_\infty,$$

giving

$$\begin{aligned}&\psi_{1,k}^{[i]-1}\circ\delta_k^{[i]'}(\nu_k)\cdot\delta_k^{[i]'}(\nu_k) \leqslant \\ &\frac{1}{2}(\epsilon_{i+1}-\epsilon_i)\delta_k^{[i]'}(\nu_k)\lambda_k^{[i]}(\nu_k)\omega_k^{*-1}\alpha_k(\nu_k) \leqslant \\ &\frac{1}{2}(\epsilon_{i+1}-\epsilon_i)\delta_i^{[i]}(\nu_k)\lambda_k^{[i]}(\nu_k)\omega_k^{*-1}\alpha_k(\nu_k).\end{aligned}$$

Third, for $k \in \boldsymbol{n}_b$, since $\gamma_{l,k} \in \mathcal{K}^o\cap\mathcal{O}(\alpha_k)$, $3i+1 \leqslant l \leqslant 3i+3$ with $\alpha_k \in \mathcal{K}^o$, by [17, Lemma 3.1], there exists $\delta_k^{[i]*} > 0$ such that

$$\begin{aligned}&\lambda^*_{3i+1}\gamma_{3i+1,k}(\nu_k) + \psi_2^{[i]-1}\circ\gamma_{3i+1,k}(\nu_k)\cdot\gamma_{3i+1,k}(\nu_k) + \\ &\lambda^*_{3i+2}\gamma_{3i+2,k}(\nu_k) + \gamma_{3i+3,k}(\nu_k) \leqslant \\ &(\epsilon_{i+1}-\epsilon_i)\delta_k^{[i]*}\lambda_k^*(\nu_k)\omega_k^{*-1}\alpha_k(\nu_k).\end{aligned}$$

Finally, by combining the above inequalities, the proof of (B10a) is complete. □

Appendix C Proof of Proposition 1

Consider system (15). Define the following new coordinate $\tilde{x} = [\tilde{x}_1^{\mathrm{T}} \ \cdots \ \tilde{x}_n^{\mathrm{T}}]^{\mathrm{T}}$ where

$$\begin{cases}\tilde{x}_1 = x_1,\ \tilde{x}_{i+1} = x_{i+1} - \rho_i(\tilde{x}_i),\ 1 \leqslant i \leqslant n-1,\\ \rho_i(\tilde{x}_i) := [\rho_{i,1}(\tilde{x}_{i,1}) \ \cdots \ \rho_{i,N}(\tilde{x}_{i,N})]^{\mathrm{T}},\ 1 \leqslant i \leqslant n,\\ \rho_{i,k}(\tilde{x}_i) = -\bar{\rho}_{i,k}(\tilde{x}_{i,k})\tilde{x}_{i,k},\ 1 \leqslant k \leqslant N,\end{cases} \tag{C1}$$

where, the function $\bar{\rho}_{i,k}(\tilde{x}_{i,k}) \geqslant 1$ is smooth, even (i.e., $\bar{\rho}_{i,k}(s) = \bar{\rho}_{i,k}(-s)$), and moreover, increasing over $[0, +\infty)$.

By (C1), system (15) can be transformed into

$$\begin{cases}\dot{z}_i = \tilde{f}_i^a(z_{[i]}, \zeta_{[i]}, \tilde{x}_{[i]}, \mu),\\ \dot{\zeta}_i = \tilde{f}_i^b(z_{[i]}, \zeta_{[i]}, \tilde{x}_{[i]}, \mu),\\ \dot{\tilde{x}}_i = -H_i\rho_i(\tilde{x}_i) + \tilde{f}_i^c(z_{[i]}, \zeta_{[i]}, \tilde{x}_{[i+1]}, \mu),\ 1 \leqslant i \leqslant n,\end{cases} \tag{C2}$$

where

$$\begin{aligned}\tilde{f}_i^a &= f_i^a(z_{[i]}, \zeta_{[i]}, \tilde{x}_1, \cdots, \tilde{x}_i + \bar{\rho}_{i-1}(\tilde{x}_{i-1}), \mu),\\ \tilde{f}_i^b &= f_i^b(z_{[i]}, \zeta_{[i]}, \tilde{x}_1, \cdots, \tilde{x}_i + \bar{\rho}_{i-1}(\tilde{x}_{i-1}), \mu),\\ \tilde{f}_1^c &= f_1^c(z_1, \zeta_1, \tilde{x}_1, \mu) + H_1\tilde{x}_2,\\ \tilde{f}_i^c &= f_i^c(z_{[i]}, \zeta_{[i]}, \tilde{x}_1, \cdots, \tilde{x}_i + \rho_{i-1}(\tilde{x}_{i-1}), \mu) + \\ &\quad H_i\tilde{x}_{i+1} - \frac{\partial\rho_{i-1}}{\partial\tilde{x}_{i-1}}(-H_{i-1}\rho_{i-1}(\tilde{x}_{i-1}) + \tilde{f}_{i-1}^c).\end{aligned}$$

Note that the origin is an equilibrium of (C2). In the following, we show the proof by Lemma 1. In other word, we need to design suitable functions $\bar{\rho}_i$ for $1 \leqslant i \leqslant n$ such that all the conditions of Lemma 1 are satisfied. This will be done in the following three steps.

First, consider (z_i, ζ_i) subsystem. Note that, by [27, Lemma A.1], there exists a function $\varrho_i \in \mathcal{K}\cap\mathcal{O}(s)$ such that

$$\|\rho_i(\tilde{x}_i)\|^2 \leqslant \varrho_i(\|\tilde{x}_i\|^2). \tag{C3}$$

By further using the inequality $\alpha(a+b) \leqslant \alpha(2a) + \alpha(2b)$ for $\alpha \in \mathcal{K}$ and $a, b \in \mathbb{R}_+$, it gives rise to

$$\begin{aligned}\gamma_{i,j}^c(\|x_j\|^2) = &\ \gamma_{i,j}^c(\|\tilde{x}_j + \rho_{j-1}(\tilde{x}_{j-1})\|^2) \leqslant \\ &\ \gamma_{i,j}^c(4\|\tilde{x}_j\|^2) + \gamma_{i,j}^c(4\|\rho_{j-1}(\tilde{x}_{j-1})\|^2) \leqslant \\ &\ \gamma_{i,j}^c(4\|\tilde{x}_j\|^2) + \gamma_{i,j}^c(4\varrho_{j-1}(\|\tilde{x}_{j-1}\|^2)).\end{aligned}$$

Then, by (21), we have

$$\dot{V}_i^a \leqslant \sum_{j=1}^{i}[\gamma_{i,j}^a(V_j^a) + \gamma_{i,j}^b(V_j^b) + \bar{\gamma}_{i,j}^c(\|\tilde{x}_j\|^2)], \tag{C4}$$

with

$$\bar{\gamma}_{i,j}^c(s) = \gamma_{i,j}^c(4s) + \gamma_{i,j+1}^c(4\varrho_j(s)) \in \mathcal{K}\cap\mathcal{O}(Id).$$

Moreover, in the same manner, by (21), we also have

$$\dot{V}_i^b \leqslant \sum_{j=1}^{i}[\delta_{i,j}^a(V_j^a) + \delta_{i,j}^b(V_j^b) + \bar{\delta}_{i,j}^c(\|\tilde{x}_j\|^2)], \tag{C5}$$

where $\bar{\delta}_{i,j}^c \in \mathcal{K}\cap\mathcal{O}(Id)$.

Second, let us consider $\tilde{x}_i$ subsystem to show the fact that function $\tilde{f}_i^c(z_{[i]}, \zeta_{[i]}, \tilde{x}_{[i+1]}, \mu)$ satisfies

$$\|\tilde{f}_i^c\|^2 \leqslant \sum_{j=1}^{i}[\bar{\psi}_{i,j}^a(V_j^a) + \bar{\psi}_{i,j}^b(V_j^b)] + \sum_{j=1}^{i+1}\bar{\psi}_{i,j}^c(\|\tilde{x}_j\|^2), \tag{C6}$$

where $\bar{\psi}_{i,j}^a \in \mathcal{K}\cap\mathcal{O}(\alpha_j)$, $\bar{\psi}_{i,j}^b \in \mathcal{K}^o\cap\mathcal{O}(\alpha_j)$ for $1 \leqslant j \leqslant i$ and $\bar{\psi}_{i,j}^c \in \mathcal{K}\cap\mathcal{O}(s)$ for $1 \leqslant j \leqslant i+1$. This will be shown by mathematical induction.

Initial Step. At the initial step $i = 1$, by (22), we have

$$\|\tilde{f}_1^c\|^2 \leqslant 2\|\tilde{f}_1^c(z_1, \zeta_1, x_1, \mu)\|^2 + 2\|H_1\|^2\|\tilde{x}_2\|^2,$$

which verifies (C6) for $i = 1$ with $\bar{\psi}_{1,1}^a = \psi_{1,1}^a$, $\bar{\psi}_{1,1}^b = \psi_{1,1}^b$, $\bar{\psi}_{1,1}^c = \psi_{1,1}^a$, $\bar{\psi}_{1,2}^c(s) = 2\|H_1\|^2 s$.

Induction Step. Suppose $i > 1$ and at step $i-1$, there exists the function $\tilde{f}_{i-1}^c(z_{[i-1]}, \zeta_{[i-1]}, \tilde{x}_{[i]}, \mu)$ satisfying (C6).

Then, it will be shown that the function $\tilde{f}_i^c(z_{[i]}, \zeta_{[i]}, \tilde{x}_{[i+1]}, \mu)$ also satisfies (C6). Toward this end, note that,

$$\begin{aligned}\|\tilde{f}_i^c\|^2 \leqslant\ & 3\|f_i^c(z_{[i]}, \zeta_{[i]}, \tilde{x}_1, \cdots, \tilde{x}_i + \rho_{i-1}(\tilde{x}_{i-1}), \mu)\|^2 + \\ & 3\|H_i\|^2\|\tilde{x}_{i+1}\|^2 + \\ & 3\|\frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(-H_{i-1}\rho_{i-1}(\tilde{x}_{i-1}) + \tilde{f}_{i-1}^c)\|^2.\end{aligned}$$

By (C3), it gives rise to

$$\begin{aligned}\|f_i^c\|^2 \leqslant\ & \sum_{j=1}^{i}[\psi_{i,j}^a(V_j^a) + \psi_{i,j}^b(V_j^b) + \psi_{i,j}^c(\|x_j\|^2)] \leqslant \\ & \sum_{j=1}^{i}[\psi_{i,j}^a(V_j^a) + \psi_{i,j}^b(V_j^b) + \psi_{i,j}^c(4\|\tilde{x}_j\|^2)] + \\ & \sum_{j=1}^{i-1}\psi_{i,j+1}^c(4\varrho_j(\|\tilde{x}_j\|^2)). \end{aligned} \tag{C7}$$

On the other hand, note that by [27, Lemma A.1], there exists a function $\bar{\varrho}_i \in \mathcal{K} \cap \mathcal{O}(s)$ such that

$$\|\frac{\partial \rho_i}{\partial \tilde{x}_i}(x_i) - \frac{\partial \rho_i}{\partial \tilde{x}_i}(0)\|^2 \leqslant \bar{\varrho}_i(\|\tilde{x}_i\|^2). \tag{C8}$$

Then, we have

$$\begin{aligned}& \|\frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(-H_{i-1}\rho_{i-1}(\tilde{x}_{i-1}) + \tilde{f}_{i-1}^c)\|^2 \leqslant \\ & \frac{1}{2}\|\frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(x_{i-1}) - \frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(0)\|^4 + \\ & \frac{1}{2}\| - H_{i-1}\rho_{i-1}(\tilde{x}_{i-1}) + \tilde{f}_{i-1}^c\|^4 + \\ & \|\frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(0)\|^2\| - H_{i-1}\rho_{i-1}(\tilde{x}_{i-1}) + \tilde{f}_{i-1}^c\|^2 \leqslant \\ & \frac{1}{2}\bar{\varrho}_i^2(\|\tilde{x}_i\|^2) + 4\|H_{i-1}\|^4\varrho_{i-1}^2(\|\tilde{x}_{i-1}\|^2) + \\ & (3i-2)[\sum_{j=1}^{i-1}[\bar{\psi}_{i,j}^{a2}(V_j^a) + \bar{\psi}_{i,j}^{b2}(V_j^b)] + \sum_{j=1}^{i}\bar{\psi}_{i,j}^{c2}(\|\tilde{x}_j\|^2)] + \\ & 2\|\frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(0)\|^2\|H_{i-1}\|^2\varrho_j(\|\tilde{x}_j\|^2) + \\ & 2\|\frac{\partial \rho_{i-1}}{\partial \tilde{x}_{i-1}}(0)\|^2\sum_{j=1}^{i-1}[\bar{\psi}_{i,j}^a(V_j^a) + \bar{\psi}_{i,j}^b(V_j^b)] + \\ & \sum_{j=1}^{i}\bar{\psi}_{i,j}^c(\|\tilde{x}_j\|^2)].\end{aligned} \tag{C9}$$

Consequently by combining (C7) and (C9), the inequality (C6) can be verified for the function $\tilde{f}_i^c(z_{[i]}, \zeta_{[i]}, \tilde{x}_{[i+1]}, \mu)$.

Third, define a positive definite quadratic function $V_i^c := V_i^c(\tilde{x}_i) = \tilde{x}_i^{\mathrm{T}}H_i^{-1}\tilde{x}_i$. By using (C6), it can be verified that, along trajectories of (C2),

$$\begin{aligned}\dot{V}_i^c \leqslant\ & -2\sum_{k=1}^{N}\bar{\rho}_{i,k}(\tilde{x}_{i,k})\tilde{x}_{i,k}^2 + \|H_i^{-1}\|^2\|\tilde{x}_i\|^2 + \\ & \sum_{j=1}^{i}[\bar{\psi}_{i,j}^a(V_j^a) + \bar{\psi}_{i,j}^b(V_j^b) + \bar{\psi}_{i,j}^c(\|\tilde{x}_j\|^2)].\end{aligned}$$

For any $\alpha_i^c \in \mathcal{K} \cap \mathcal{O}(Id)$, noting that $\bar{\psi}_{i,i}^c \in \mathcal{K} \cap \mathcal{O}(Id)$, by [19, Lemma 7.8], there exists an even function $\bar{\rho}_{i,k}^* \geqslant 1$ that is increasing over $[0, +\infty)$, such that

$$\begin{aligned}& \bar{\psi}_{i,i}^c(\|\tilde{x}_i\|^2) + \alpha_i^c(\tilde{x}_i^{\mathrm{T}}H_i^{-1}\tilde{x}_i) + \|H_i^{-1}\|^2\|\tilde{x}_i\|^2 \leqslant \\ & \sum_{k=1}^{N}\bar{\rho}_{i,k}^*(\tilde{x}_{i,k})\tilde{x}_{i,k}^2.\end{aligned}$$

Then, choosing

$$\bar{\rho}_{i,k}(\tilde{x}_{i,k}) \geqslant \bar{\rho}_{i,k}^*(\tilde{x}_{i,k}),\ 1 \leqslant k \leqslant N,\ 1 \leqslant i \leqslant n, \tag{C10}$$

leads to

$$\dot{V}_i^c \leqslant \sum_{j=1}^{i}\phi_{i,j}^a(V_j^a) + \sum_{j=1}^{i}\phi_{i,j}^b(V_j^b) + \sum_{j=1}^{i+1}\phi_{i,j}^c(V_j^c), \tag{C11}$$

for $V_{n+1}^c \equiv 0$, and functions $\phi_{i,j}^a = \bar{\psi}_{i,j}^a \in \mathcal{K} \cap \mathcal{O}(\alpha_j^a)$, $\phi_{i,j}^b\bar{\psi}_{i,j}^b \in \mathcal{K}^o \cap \mathcal{O}(\alpha_j^b)$, $\phi_{i,j}^c\bar{\psi}_{i,j}^c \in \mathcal{K} \cap \mathcal{O}(Id)$, where in particular, $\phi_{i,i}^c(V_i^c) = -\alpha_i^c(V_i^c)$.

Hence, Assumption 1 is verifiable. Moreover, by Lemma 1, one can construct gain functions $\alpha_i^c \in \mathcal{K} \cap \mathcal{O}(Id)$ for $1 \leqslant i \leqslant n$ such that system (C2) is globally asymptotically stable at the origin. Furthermore, such gain functions can be specified by designing functions $\bar{\rho}_i$ for $1 \leqslant i \leqslant n$ in (C10). The proof is complete.

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