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状态和输入受限的切换奇异布尔控制网络的最优控制

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摘要: 本文研究了状态和输入均受限的切换奇异布尔控制网络的最优控制问题. 利用矩阵半张量积方法获得受限切换奇异布尔控制网络的等价代数形式. 然后通过类似针变化得到了存在最优控制的必要条件, 并且提出了一个算法设计切换序列和控制策略使收益函数最大化. 最后给出例子验证所得结果的有效性.

关键词: 切换奇异布尔控制网络; 状态和输入受限; 最优控制; 半张量积

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Optimal control of switched singular Boolean control networks with state and input constraints

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Abstract: In the present paper, the optimal control problem of switched singular Boolean control networks (SSBCNs) with state and input constraints is investigated. By using the semi-tenser product of matrices, the parallel constrained algebraic form is obtained for constrained SSBCNs. Then a necessary condition for the existence of optimal control is presented by using an analogous needle variation. An algorithm is proposed to design the proper switching sequence and control strategy which maximizes the cost functional at a fixed termination time. Finally, a numerical example is given to show that the new results obtained in this paper are very effective.

Key words: switched singular Boolean control network; state and input constraints; optimal control; semi-tensor product

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1 Introduction

The study of genetic regulatory networks has become an important part of the biological system, because it has long been the desire to explore the mysteries of living organisms. Many kinds of computational models have been proposed to simulate the reproduction of genetic regulatory networks, including ordinary differential equations[1], Boolean networks (BNs)[2], Bayesian networks[3], Neural networks[4] and so on. Among these models, BN which was firstly introduced by Kauffman in 1969 has received the most wide applications as an effective tool for analyzing genetic regulatory networks. In a BN, the state of a gene is quantized into only two levels (active: 1 or inactive: 0), and the expression level of a given gene can be obtained by using the Boolean function to a plurality of related gene expression levels. Though BN is a simplified model, it becomes a powerful tool in analysing genetic regulatory networks. And some significant results were presented[5–6].

In recent years, a new matrix product, namely, the semi-tensor product (STP) of matrices, has been presented by Cheng[7], which is a generalization of the tradition matrix product, and has been successfully applied to convert a logical function into an algebraic form. Using this new mathematical tool, numerous control problems about BNs were investigated, such as controllability and observability problems[8], synchronization problem[9], consistent stabilizability problem[10], disturbance decoupling problem[11], and so on. Optimal control is one of the fundamental concepts and research topics in control theory. Using the above STP-based framework, Refs. [12–13] discussed the Pontryagin maximum principle for the Mayer-type optimal con-

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trol of BCNs. Ref. [14] studied optimal infinite-horizon control problem, motivated by a finite strategy game between human and machine.

Singular Boolean networks, which are also referred to as dynamic-algebraic Boolean networks, have attracted much attention in the past several decades due to the fact that they are much more convenient and effective than standard models to describe many science and engineering systems, including biological systems, aircraft attitude control, power systems, and social economic systems[15]. In Ref. [16], the fundamental problems for singular Boolean networks were discussed by Feng et al, including the condensed algebraic expressions, normalisation problem, solvability and limit sets, which make an important contribution to the further research on singular Boolean networks. Some other conclusions about Singular Boolean networks can be found in [17–19]. It is noticed that switched systems play a crucial role in the study of control theory. In practice, many biological systems appear with different model structures according to the environment changes. A practical example is the genetic switch in the bacteriophage λ, which contains two different models: lysis and lysogeny[20]. When modeling biological systems as Boolean networks, the dynamics becomes switched Boolean networks (SBNs), which is governed by different Boolean dynamic models. There have been some recent results about SBNs. For example, Ref. [21] considered the time-optimal state feedback stabilization of SBCNs, and an algorithm for finding all time-optimal switching state feedbacks was proposed. Ref. [22] studied the complete synchronization problem for the drive-response SBNs, and some necessary and sufficient conditions were presented. The output tracking problem of SBNs was discussed in Ref. [23], and a novel design procedure was established. Ref. [24] investigated the set stability of SBNs, and a necessary and sufficient condition for set stability was obtained. On the other hand, it is well known that some states and inputs of biological systems are actually undesirable ones because they correspond to unfavorable situations. For instance, the state "Wnt5a=1" of the WNT5A gene regulatory network is undesirable because it may lead to the possibility of cancer metastasis increased[25]. Hence, it is necessary to put some constraints to the undesirable states and inputs in biological systems. In Ref. [26], the authors firstly developed a BN with constraint states and gained some interesting results. The controllability and stabilization of SBCNs with state and input constraints were considered in Ref. [27]. Vast results on SBCNs and singular Boolean control networks have been obtained, respectively. There are, to the best of our knowledge, no results on the study of SSBCNs with state and input constraints which are also called the constrained ones. The presence of state and input constraints makes the analysis of SSBCNs much more complicated. Furthermore, the results obtained for unconstrained SSBCNs can hardly be applied to constrained ones. Therefore, either in theory or in practice, it is significant and necessary to study SSBCNs with state and input constraints. This motivates the present work of this paper.

In this paper, using the STP-based framework, we investigate the optimal control problem of SSBCNs with state and input constraints. Firstly, we propose a necessary and sufficient condition for the uniqueness of solution of a SSBCN under any switching signal and any control, and convert a SSBCN into an equivalent SBCN. Secondly, we consider a constrained SBCN and convert it into an equivalent constrained algebraic form. Finally, according to the parallel constrained algebraic form, we obtain a necessary condition for the existence of optimal control of the constrained SSBCN.

The remainder of this paper is organized as follows: Section 2 introduces some preliminaries about STP. In Section 3, we investigate the optimal control problem of SSBCNs with state and input constraints and present the main results of this paper. Section 4 shows an example to illustrate the main results obtained in this paper. Finally, a brief summary is given in Section 5.

2 Preliminaries

In this section, we introduce the semi-tensor product of matrices and the matrix expression of logic, which are summary mainly from Ref. [7].

2.1 Semi-tensor product of matrices

Definition 1[7] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product of A and B is defined as $A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p})$, where $\otimes$ is the Kronecker product.

Remark 1 If $n = p$, the STP of matrices becomes conventional matrix product. Thus, all the fundamental properties of conventional matrix product remain true. Based on this, we can omit the symbol $\ltimes$, if no confusion raises.

Next, we introduce some notations, which will be used throughout this paper.

1) δ_n^i : the ith column of the identity matrix I_n.

2) $\varDelta_n := \{\delta_n^i | i = 1, 2, \cdots, n\}$. For simplicity, let $\varDelta := \varDelta_2$.

3) $\mathrm{Col}(A)(\mathrm{Row}(A))$ is the set of columns (rows) of A. $\mathrm{Col}_i(A)(\mathrm{Row}_i(A))$ is the ith column (row) of A.

4) A matrix $A \in \mathbb{R}^{m\times n}$ is called a logical matrix, if the columns of A, denoted by $\mathrm{Col}(A)$, are of the form of δ_m^k. That is, $\mathrm{Col}(A) \subset \varDelta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.

5) If $L \in \mathcal{L}_{m\times n}$, by definition it can be expressed as $L = [\delta_m^{i_1}\ \delta_m^{i_2}\ \cdots\ \delta_m^{i_n}]$, and its shorthand is $L = \delta_m[i_1\ i_2\ \cdots\ i_n]$.

6) For $A \in \mathbb{R}^{m\times r}$, $B \in \mathbb{R}^{n\times r}$,

$$A * B = [\mathrm{Col}_1(A) \ltimes \mathrm{Col}_1(B) \ \cdots \ \mathrm{Col}_r(A) \ltimes \mathrm{Col}_r(B)]$$

is the Khatri-Rao product of A and B.

7) Let $X \in \mathbb{R}^{m\times 1}$ and $Y \in \mathbb{R}^{n\times 1}$ be two column vectors. Then $Y \ltimes X = W_{[m,n]} \ltimes X \ltimes Y$, where $W_{[m,n]} \in \mathcal{L}_{mn\times mn}$ is called the swap matrix, which is given as

$$\begin{aligned} W_{[m,n]} = \delta_{mn}[& 1 \ m+1 \ \cdots \ (n-1)m+1 \\ & 2 \ m+2 \ \cdots \ (n-1)m+2 \\ & \cdots \\ & m \ m+m \ \cdots \ (n-1)m+m]. \end{aligned}$$

8) Assume $X \in \Delta_p$ and $Y \in \Delta_q$. We define two dummy matrices, named by "front-maintaining operator" and "rear-maintaining operator" respectively as

$$D_{\mathrm{f}}^{p,q} = \delta_p[\underbrace{1\cdots 1}_{q}\ \underbrace{2\cdots 2}_{q}\cdots\underbrace{p\cdots p}_{q}],$$

$$D_{\mathrm{r}}^{p,q} = \delta_q[\underbrace{\underbrace{1\,2\cdots q}\ \underbrace{1\,2\cdots q}\cdots\underbrace{1\,2\cdots q}}_{p}].$$

Then we have $D_{\mathrm{f}}^{p,q}XY = X,\ D_{\mathrm{r}}^{p,q}XY = Y$.

2.2 Matrix expression of logic

In this subsection, we recall the vector form of Boolean variables and the matrix expression of logic. Using the semi-tensor product of matrices, a logical function can be converted into an algebraic function. To do this, we give logical values a vector form as follows: $\mathcal{D} := \{0,1\}$, where $1 \sim T$ means "true" and $0 \sim F$ means "false". Then the logical variable $x(t)$ takes value from $\mathcal{D}$, expressed as $x(t) \in \mathcal{D}$. Identifying $T = 1 \sim \delta_2^1,\ F = 0 \sim \delta_2^2$, according to the variable types, the "$\mathcal{D}$" and "Δ" can be used freely, i.e.

$$x(t) \in \Delta := \Delta_2 = \{\delta_2^1, \delta_2^2\}.$$

Next, we give a lemma that is fundamental for the matrix expression of logical functions.

Lemma 1[7] Any logical function $f(x_1,\cdots,x_n)$ with logical arguments $x_1,\cdots,x_n \in \Delta$ can be expressed in a multi-linear form as $f(x_1,\cdots,x_n) = M_{\mathrm{f}}x_1,\cdots,x_n$, where $M_{\mathrm{f}} \in \mathcal{L}_{2\times 2^n}$ is unique, called the structure matrix of logical function f.

To see the results of the structure matrix, please refer to Ref. [8] for details.

In the end, we give a lemma, which will be used in the sequel.

Lemma 2[27] For any integer $i \in \{1,\cdots,\omega\beta\}$, there exist unique positive integers i_1 and i_2 such that

$$\delta_{\omega\beta}^i = \delta_\omega^{i_1} \ltimes \delta_\beta^{i_2}, \tag{1}$$

where

$$i_1 = \begin{cases} k, & i = km_1,\ k = 1,\cdots,\omega; \\ [\dfrac{i}{m_1}]+1, & \text{otherwise}. \end{cases} \tag{2}$$

$[\dfrac{i}{\beta}]$ denotes the larger less than or equal to $\dfrac{i}{\beta}$, and $i_2 = i-(i_1-1)\beta$.

3 Main results

In this section, the main results of this paper are presented. First the SSBCN is converted into an equivalent SBCN. Then consider a constrained SBCN and convert it into an equivalent constrained algebraic form. Finally, based on the parallel constrained algebraic form, we obtain a necessary condition for the existence of optimal control of the constrained SSBCN.

3.1 Constrained algebraic form

Consider the following switched Boolean control network with n nodes, m control inputs and ω subnetworks:

$$\begin{cases} g_1^{\sigma(t)}(X(t+1)) = f_1^{\sigma(t)}(X(t),U(t)), \\ g_2^{\sigma(t)}(X(t+1)) = f_2^{\sigma(t)}(X(t),U(t)), \\ \quad\vdots \\ g_n^{\sigma(t)}(X(t+1)) = f_n^{\sigma(t)}(X(t),U(t)), \end{cases} \tag{3}$$

where $\sigma: \mathbb{N} \to \Omega = \{1,2,\cdots,\omega\}$ is the switching signal, $X(t) = (x_1(t),x_2(t),\cdots,x_n(t)) \in \mathcal{D}^n$ is the logical state, $U(t) = (u_1(t),u_2(t),\cdots,u_m(t)) \in \mathcal{D}^m$ is the logical input, and $f_i^j: \mathcal{D}^{n+m} \to \mathcal{D},\ g_i^j: \mathcal{D}^n \to \mathcal{D},\ i = 1,2,\cdots,n,\ j = 1,2,\cdots,\omega$ are logic functions.

Assume that the structure matrices of $g_i^{\sigma(t)}$ and $f_i^{\sigma(t)}$ are $G_i^{\sigma(t)}$ and $M_i^{\sigma(t)}$, respectively. By setting $x(t) = \ltimes_{i=1}^n x_i(t),\ u(t) = \ltimes_{i=1}^m u_i(t)$, using Lemma 1, we obtain

$$\begin{cases} G_1^{\sigma(t)}x(t+1) = M_1^{\sigma(t)}u(t)x(t), \\ G_2^{\sigma(t)}x(t+1)) = M_2^{\sigma(t)}u(t)x(t), \\ \quad\vdots \\ G_n^{\sigma(t)}x(t+1) = M_n^{\sigma(t)}u(t)x(t), \end{cases} \tag{4}$$

where $G_i^{\sigma(t)} \in \mathcal{L}_{2\times 2^n}$ and $M_i^{\sigma(t)} \in \mathcal{L}_{2\times 2^{n+m}},\ i = 1,2,\cdots,n$ are uniquely determined by $g_i^{\sigma(t)}$ and $f_i^{\sigma(t)}$, respectively. Multiplying both sides of Eqs. (4), yields

$$E_{\sigma(t)}x(t+1) = F_{\sigma(t)}u(t)x(t), \tag{5}$$

where $E_{\sigma(t)} \in \mathcal{L}_{2^n\times 2^n},\ F_{\sigma(t)} \in \mathcal{L}_{2^n\times 2^{n+m}}$,

$$E_{\sigma(t)} = G_1^{\sigma(t)} * G_2^{\sigma(t)} * \cdots * G_n^{\sigma(t)}$$

and

$$F_{\sigma(t)} = M_1^{\sigma(t)} * M_2^{\sigma(t)} * \cdots * M_n^{\sigma(t)}.$$

When $\mathrm{rank}(E_i) < 2^n,\ \forall i \in \Omega$, the SBCN (3) is called a switched singular Boolean control network. In this condition, the SSBCN (5) and the general switched singular systems[15] have the same form. In this paper, we assume that $\mathrm{rank}(E_i) < 2^n,\ \forall i \in \Omega$.

Because the solution of the SSBCN (5) may not be unique just like the ordinary switched singular systems for any initial value, we give a necessary and sufficient condition for the uniqueness of solution of the SSBCN (5) under any switching signal and any control.

Lemma 3[16] Singular Boolean network $\hat{E}x(t+1) = \hat{F}x(t)$ has a unique solution for any initial value, if and only if $\mathrm{Col}(\hat{F}) \subseteq \mathrm{Col}(\hat{E})$ and there is only one integer $j \in \{1, 2, \cdots, 2^n\}$ satisfying $\mathrm{Col}_j(\hat{E}) = \mathrm{Col}_i(\hat{F})$ for every $i \in \{1, 2, \cdots, 2^n\}$.

Consider SSBCN (5) again. Split $F_{\sigma(t)} \in \mathcal{L}_{2^n \times 2^{n+m}}$ into 2^m equal-size blocks as

$$F_{\sigma(t)} = [\mathrm{Blk}_1(F_{\sigma(t)})\ \mathrm{Blk}_2(F_{\sigma(t)})\ \cdots\ \mathrm{Blk}_{2^m}(F_{\sigma(t)})],$$

where $\mathrm{Blk}_i(F_{\sigma(t)}) \in \mathcal{L}_{2^n \times 2^n},\ i \in \{1, 2, \cdots, 2^m\}$. Then, we have the following solvability result.

Lemma 4 Consider the SSBCN (5). Under any switching signal $\sigma(t)$ and any control $u(t)$, the solution of the SSBCN (5) is unique for any initial value, if and only if the following two conditions hold:

A1) $\mathrm{Col}(F_i) \subseteq \mathrm{Col}(E_i),\ \forall i \in \Omega$;

A2) there is only one integer $j \in \{1, 2, \cdots, 2^n\}$ satisfying $\mathrm{Col}_j(E_i) = \mathrm{Col}_h(F_i),\ \forall i \in \Omega,\ \forall h \in \{1, 2, \cdots, 2^{n+m}\}$.

Proof Set $\sigma(t) = i \in \Omega,\ u(t) = \delta_{2^m}^k$, then the SSBCN (5) becomes singular Boolean network $E_i x(t+1) = F_{i_k} x(t)$, where F_{i_k} denotes the kth block of the matrix F_i. Thus, Under any switching signal $\sigma(t)$ and any control $u(t)$, the solution of the SSBCN (5) is unique for any initial value if and only if for every $i \in \Omega$ and every $k \in \{1, 2, \cdots, 2^m\}$, singular Boolean network $E_i x(t+1) = F_{i_k} x(t)$, has a unique solution for any initial value. Based on Proposition 1, we know that under any switching signal $\sigma(t)$ and any control $u(t)$, the solution of the SSBCN (5) is unique for any initial value if and only for every $i \in \Omega$, every $k \in \{1, 2, \cdots, 2^m\}$, i) $\mathrm{Col}(F_{i_k}) \subseteq \mathrm{Col}(E_i)$, and ii) there is only one integer $j \in \{1, 2, \cdots, 2^n\}$ satisfying $\mathrm{Col}_j(E_i) = \mathrm{Col}_h(F_{i_k})$ for every $h \in \{1, 2, \cdots, 2^n\}$. It is equivalent to that $\mathrm{Col}(F_i) \subseteq \mathrm{Col}(E_i),\ \forall i \in \Omega$, and there is only one integer $j \in \{1, 2, \cdots, 2^n\}$ satisfying $\mathrm{Col}_j(E_i) = \mathrm{Col}_h(F_i),\ \forall i \in \Omega,\ \forall h \in \{1, 2, \cdots, 2^{n+m}\}$. QED.

In the following, we always assume that Lemma 4 holds.

For ease of research, we first convert the SSBCN (5) into an equivalent SBCN.

For each $i \in \Omega$, we define a matrix $L_i \in \mathcal{L}_{2^n \times 2^{n+m}}$ as

$$\mathrm{Col}_j(L_i) = \delta_{2^n}^{r_j^i},\ \text{if}\ \mathrm{Col}_j(F_i) = \mathrm{Col}_{r_j^i}(E_i),\quad j = 1, 2, \cdots, 2^{n+m}. \tag{6}$$

Then, we can obtain the following conclusion.

Theorem 1 Assume that Lemma 4 holds. The SSBCN (5) is equal to the following SBCN:

$$x(t+1) = L_{\sigma(t)} u(t) x(t). \tag{7}$$

Proof For any initial value $x(0)$, any switching signal $\sigma(t)$ and any control $u(t)$, let $\hat{x}(t) = \hat{x}(t; x(0), \sigma, u)$ be the solution of the SSBCN (5), and $x(t) = x(t; x(0), \sigma, u)$ be the solution of the SBCN (7). We need to prove that

$$\hat{x}(t) = x(t),\ \forall t \in \mathbb{Z}^+. \tag{8}$$

By induction on t, assume $\sigma(0) = \delta_\omega^{i_0},\ u(0)x(0) = \delta_{2^{m+n}}^{j_0}$. We first consider $t = 1$. For one thing, with simple calculation,

$$x(1) = L_{\sigma(0)} u(0) x(0) = \mathrm{Col}_{j_0}(L_{i_0}) = \delta_{2^n}^{r_{j_0}^{i_0}}.$$

For another, since

$$E_{\sigma(0)} \hat{x}(1) = F_{\sigma(0)} u(0) x(0) = \mathrm{Col}_{j_0}(F_{i_0}) = \mathrm{Col}_{r_{j_0}^{i_0}}(E_{i_0}),$$

we obtain $\hat{x}(1) = \delta_{2^n}^{r_{j_0}^{i_0}}$. This proves (8) for $t = 1$.

Assume that the result holds for $t = k$. Setting $\sigma(k) = \delta_\omega^{i_1}$ and $u(k)\hat{x}(k) = u(k)x(k) = \delta_{2^{m+n}}^{j_1}$, we now consider the case of $t = k+1$. For the SSBCN (5), since

$$E_{\sigma(k)} \hat{x}(k+1) = F_{\sigma(k)} u(k) \hat{x}(k) = \mathrm{Col}_{j_1}(F_{i_1}) = \mathrm{Col}_{r_{j_1}^{i_1}}(E_{i_1}),$$

we obtain $\hat{x}(k+1) = \delta_{2^n}^{r_{j_1}^{i_1}}$. For the SBCN (7), a simple calculation shows that

$$x(k+1) = L_{\sigma(k)} u(k) x(k) = \mathrm{Col}_{j_1}(L_{i_1}) = \delta_{2^n}^{r_{j_1}^{i_1}},$$

which means that $\hat{x}(k+1) = x(k+1)$.

By induction, (8) holds for any $t \in \mathbb{Z}^+$. QED.

It is noted that some states and inputs may correspond to unfavorable or dangerous situations in biological systems, thus, we need to put some constraints to these undesirable states and inputs. We now consider the SBCN (7) with state and input constraints, i.e. the constrained SBCN (7).

For any $t \in \mathbb{N}$, assume that $x(t) \in C_{\mathrm{x}} \subseteq \Delta_{2^n}$ and $u(t) \in C_{\mathrm{u}} \subseteq \Delta_{2^m}$, where C_{x} with $1 \leqslant |C_{\mathrm{x}}| \leqslant 2^n$ denotes the state's constraint set, C_{u} with $1 \leqslant |C_{\mathrm{u}}| \leqslant 2^m$ denotes the input's constraint set, and $|C_{\mathrm{x}}|$ and $|C_{\mathrm{u}}|$ stand for the cardinalities of the sets C_{x} and C_{u}, respectively. Set $|C_{\mathrm{x}}| = \alpha$ and $|C_{\mathrm{u}}| = \beta$, then C_{x} and C_{u} can be expressed as

$$\begin{cases} C_{\mathrm{x}} = \{\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \cdots, \delta_{2^n}^{i_\alpha},\ i_1 < i_2 < \cdots < i_\alpha\}, \\ C_{\mathrm{u}} = \{\delta_{2^m}^{j_1}, \delta_{2^m}^{j_2}, \cdots, \delta_{2^m}^{j_\beta},\ j_1 < j_2 < \cdots < j_\beta\}. \end{cases} \tag{9}$$

Next, the constrained SBCN (7) is converted into an equivalent constrained algebraic form.

Construct the following block selection matrices:

$$J_i^{(p,q)} := \underbrace{[0_{q\times q} \ \cdots \ 0_{q\times q} \ \underbrace{I_q}_{i\text{th}} \ 0_{q\times q} \ \cdots \ 0_{q\times q}]}_{p}, \quad (10)$$

where $J_i^{(p,q)} \in \mathbb{R}^{q\times pq}$, $i = 1, 2, \cdots, p$, $0_{q\times q}$ is the $q\times q$ zero matrix, and $I_q \in \mathcal{L}_{q\times q}$ denotes $q\times q$ identity matrix.

Lemma 5[27] 1) Given a matrix $A \in \mathbb{R}^{pq\times r}$, split A as

$$\begin{bmatrix} A_1 \\ \vdots \\ A_p \end{bmatrix},$$

where $A_i \in \mathbb{R}^{q\times r}$. Then,

$$J_i^{(p,q)} A = A_i. \quad (11)$$

2) Given a matrix $B \in \mathbb{R}^{r\times pq}$, split B as: $B = [B_1 \ \cdots \ B_p]$, where $B_i \in \mathbb{R}^{r\times q}$. Then,

$$B(J_i^{(p,q)})^{\mathrm{T}} = B_i. \quad (12)$$

According to Lemma 3.1, let

$$\varphi_{\mathrm{x}} = \begin{bmatrix} J_{i_1}^{(2^n,1)} \\ \vdots \\ J_{i_\alpha}^{(2^n,1)} \end{bmatrix}, \quad (13)$$

$$\varphi_{\mathrm{u}} = \begin{bmatrix} J_{j_1}^{(2^m,1)} \\ \vdots \\ J_{j_\beta}^{(2^m,1)} \end{bmatrix}. \quad (14)$$

Denote $\delta_\alpha^0 = 0_{\alpha\times 1}$ and $\delta_\beta^0 = 0_{\beta\times 1}$. Then, the state $x(t) \in \Delta_{2^n}$ and control $u(t) \in \Delta_{2^m}$ of the constrained SBCN (7) can be converted into the following form:

$$\begin{cases} \bar{x}(t) = \varphi_{\mathrm{x}} x(t) \in \bar{C}_{\mathrm{x}}, \\ \bar{u}(t) = \varphi_{\mathrm{u}} u(t) \in \bar{C}_{\mathrm{u}}, \end{cases} \quad (15)$$

where $\bar{C}_{\mathrm{x}} = \{\delta_\alpha^1, \cdots, \delta_\alpha^\alpha\} \cup \{\delta_\alpha^0\}$ and $\bar{C}_{\mathrm{u}} = \{\delta_\beta^1, \cdots, \delta_\beta^\beta\} \cup \{\delta_\beta^0\}$.

Consider the SBCN (7). For each $i \in \Omega$, setting $L_i = [\mathrm{Blk}_1(L_i) \ \cdots \ \mathrm{Blk}_{2^m}(L_i)]$ and using the block selection matrices, we have the following matrix

$$\bar{L}_i = [\overline{\mathrm{Blk}}_1(L_i) \ \cdots \ \overline{\mathrm{Blk}}_{2^m}(L_i)][(J_{j_1}^{(2^m,\alpha)})^{\mathrm{T}} \ \cdots \ (J_{j_\beta}^{(2^m,\alpha)})^{\mathrm{T}}] \in \mathcal{B}_{\alpha\times\alpha\beta}, \ i = 1, 2, \cdots, \omega, \quad (16)$$

where

$$\overline{\mathrm{Blk}}_s(L_i) = \begin{bmatrix} J_{i_1}^{(2^n,1)} \\ \vdots \\ J_{i_\alpha}^{(2^n,1)} \end{bmatrix} \mathrm{Blk}_s(L_i)[(J_{i_1}^{(2^n,1)})^{\mathrm{T}} \ \cdots \ (J_{i_\alpha}^{(2^n,1)})^{\mathrm{T}}] \in \mathcal{B}_{\alpha\times\alpha}, \ s = 1, 2, \cdots, 2^m. \quad (17)$$

Identifying the switching signal $\sigma = i \sim \delta_\omega^i \in \Delta_\omega$, $i \in \Omega$, and defining $\bar{L} = [\bar{L}_1 \ \cdots \ \bar{L}_\omega]$, by the above transformation, the SBCN (7) can be converted into the following form:

$$\bar{x}(t+1) = \bar{L}\sigma(t)\bar{u}(t)\bar{x}(t), \quad (18)$$

where $\bar{L} \in \mathcal{B}_{\alpha\times\omega\alpha\beta}$.

Theorem 2 The state trajectories of the SBCN (7) with C_{x} and C_{u} are equal to those of the SBCN (18) with $\bar{C}_{\mathrm{x}}$ and $\bar{C}_{\mathrm{u}}$.

Proof Let $\bar{x}(t) = \bar{x}(t; \bar{x}(0), \sigma, \bar{u})$ denote the trajectory of the SBCN (18) corresponding to initial value $\bar{x}(0) \in \bar{C}_{\mathrm{x}}$, switching signal $\sigma \in \Delta_\omega$ and control $\bar{u} \in \bar{C}_{\mathrm{u}}$. We need to prove that

$$\bar{x}(t+1) = \varphi_{\mathrm{x}} x(t+1), \ \forall t \in \mathbb{N}, \quad (19)$$

where φ_{x} is given in (13).

First, for $\forall t \in \mathbb{N}$, $\forall \sigma(t) \in \Delta_\omega$, $\forall \ u(t) \in C_{\mathrm{u}}$ and $\forall x(t) \in C_{\mathrm{x}}$, setting $\sigma(t) = \delta_\omega^r$, $u(t) = \delta_{2^m}^{j_k}$ and $x(t) = \delta_{2^n}^{i_s}$, if $x(t+1) = L_{\sigma(t)}u(t)x(t) \in C_{\mathrm{x}}$, say, $x(t+1) = \delta_{2^n}^{i_h}$, $h \in \{1, \cdots, \alpha\}$, one can easily obtain that

$$\bar{x}(t+1) = \bar{L}\sigma(t)\bar{u}(t)\bar{x}(t) = \delta_\alpha^h \in \bar{C}_{\mathrm{x}} \setminus \{\delta_\alpha^0\},$$

where $\bar{x}(t) = \delta_\alpha^s \in \bar{C}_{\mathrm{x}}$ and $\bar{u}(t) = \delta_\beta^k \in \bar{C}_{\mathrm{u}}$.

If $x(t+1) = L_{\sigma(t)}u(t)x(t) = \delta_{2^n}^{i_p} \notin C_{\mathrm{x}}$, then

$$\bar{x}(t+1) = \bar{L}\sigma(t)\bar{u}(t)\bar{x}(t) = \delta_\alpha^0.$$

Thus, in both cases, $\bar{x}(t+1) = \varphi_{\mathrm{x}} x(t+1)$. Then, for

$$\forall t \in \mathbb{N}, \ \forall \sigma(t) \in \Delta_\omega, \ \forall \bar{u}(t) \in \bar{C}_{\mathrm{u}} \setminus \{\delta_\beta^0\}$$

and $\forall \bar{x}(t) \in \bar{C}_{\mathrm{x}} \setminus \{\delta_\alpha^0\}$, setting $\sigma(t) = \delta_\omega^r$, $\bar{u}(t) = \delta_\beta^k$ and $\bar{x}(t) = \delta_\alpha^s$, if

$$\bar{x}(t+1) = \bar{L}\sigma(t)\bar{u}(t)\bar{x}(t) \in \bar{C}_{\mathrm{x}} \setminus \{\delta_\alpha^0\},$$

say, $\bar{x}(t+1) = \delta_\alpha^h$, $h \in \{1, \cdots, \alpha\}$, it is easy to achieve that $x(t+1) = L_{\sigma(t)}u(t)x(t) = \delta_{2^n}^{i_h} \in C_{\mathrm{x}}$. If $\bar{x}(t+1) = \bar{L}\sigma(t)\bar{u}(t)\bar{x}(t) = \delta_\alpha^0$, then $x(t+1) = L_{\sigma(t)}u(t)x(t) \notin C_{\mathrm{x}}$. Thus, one can know that $\bar{x}(t+1) = \varphi_{\mathrm{x}} x(t+1)$, $\forall t \in \mathbb{N}$.

Based on the above analysis, Theorem 2 holds.

QED.

Remark 2 By Theorem 1 and Theorem 2, we can obtain that the state trajectories of the SSBCN (5) with C_{x} and C_{u} are parellel to those of the SBCN (18) with $\bar{C}_{\mathrm{x}}$ and $\bar{C}_{\mathrm{u}}$. And (18) is called the parellel constrained algebraic form of the original network. Thus, we can convert the optimal control problem of the SSBCN (5) with state and input constraints into that of the SBCN (18).

Before the end of this subsection, we propose the definition of the transition matrix for the SBCN (18), which will be used in the sequel. Considering the SBCN (18), for $l \leqslant s$, we obtain

$$\bar{x}(s) = \bar{L}\sigma(s-1)\bar{u}(s-1)\cdots\bar{L}\sigma(l)\bar{u}(l)\bar{x}(l) := H(s, l; \sigma, \bar{u})\bar{x}(l), \quad (20)$$

where

$$H(s,l;\sigma,\bar{u}) = \bar{L}\sigma(s-1)\bar{u}(s-1)\cdots\bar{L}\sigma(l)\bar{u}(l) \tag{21}$$

with $H(s,l;\sigma,\bar{u}) = I_\alpha$, if $s = l$. The matrix $H(s, l;\sigma,\bar{u}) \in \mathcal{L}_{\alpha\times\alpha}$ is called the transition matrix of the SBCN (18) from time l to time s corresponding to the switching signal σ and the control $\bar{u}$. For any $l \leqslant k \leqslant s$, a simple calculation shows that

$$H(s,l;\sigma,\bar{u}) = H(s,k;\sigma,u)H(k,l;\sigma,\bar{u}). \tag{22}$$

3.2 Optimal control

In this subsection, we investigate the Mayer-type optimal control problem of the SSBCN (5) with state and input constraints and provide a necessary condition for the existence of optimal control.

Consider the SSBCN (5) with the initial state $x(0) \in C_{\mathrm{x}}$. Fix a termination time $s \geqslant 1$. Let $\pi = \{(0, \sigma(0)),\cdots,((s-1),\sigma(s-1))\}$ and $\boldsymbol{U} = \{\{u(0), \cdots, u(s-1)\} : u(t) \in C_{\mathrm{u}},\ t = 0,1,\cdots,s-1\}$ denote the sets of switching sequence and admissible control sequence, respectively. The Mayer-type optimal control problem is to find a proper switching sequence and a control strategy that maximize (or minimize) the cost functional

$$J(\sigma,u) = r^{\mathrm{T}}x(s), \tag{23}$$

where $x(t) \in C_{\mathrm{x}},\ t = 0,1,\cdots,s$, and $r = [r_1\ \cdots\ r_{2^n}]^{\mathrm{T}} \in \mathbb{R}^{2^n\times 1}$ is a given constant vector.

According to Remark 2 , we can convert the Mayer-type optimal control problem of the SSBCN (5) with state and input constraints into that of the SBCN (18). That is, the cost functional $J(\cdot)$ in (23) becomes

$$\bar{J}(\sigma,\bar{u}) = \bar{r}^{\mathrm{T}}\bar{x}(s), \tag{24}$$

where $\bar{r} = \varphi_{\mathrm{x}}r,\ \bar{x}(t) = \varphi_{\mathrm{x}}x(t) \in \bar{C}_{\mathrm{x}} \setminus \{\delta_\alpha^0\},\ \bar{u}(t) = \varphi_{\mathrm{u}}u(t) \in \bar{C}_{\mathrm{u}} \setminus \{\delta_\beta^0\},\ t = 0,1,\cdots,s$, and φ_{x} and φ_{u} are given in (13) and (14), respectively.

Remark 3 Since $x(t) \in C_{\mathrm{X}}$ and $u(t) \in C_{\mathrm{u}},\ t = 0,1,\cdots,s$, a simple calculation shows that $\bar{x}(t) = \varphi_{\mathrm{X}}x(t) \in \bar{C}_{\mathrm{x}} \setminus \{\delta_\alpha^0\}$ and $\bar{u}(t) = \varphi_{\mathrm{u}}u(t) \in \bar{C}_{\mathrm{u}} \setminus \{\delta_\beta^0\},\ t = 0,1,\cdots,s$.

Theorem 3 Consider the SBCN (18). Denote a proper switching sequence by $\pi^* = \{(0,\sigma^*(0)),\cdots, ((s-1),\sigma^*(s-1))\}$ and an optimal control sequence by $\bar{u}^* = \{\bar{u}^*(0),\ \cdots,\ \bar{u}^*(s-1)\}$. Define the adjoint $y : \{1,2,\cdots,s\} \to \mathbb{R}^\alpha$ as the solution of

$$\begin{cases} y(t) = (\bar{L}\sigma^*(t)\bar{u}^*(t))^{\mathrm{T}}y(t+1), \\ y(s) = \bar{r}, \end{cases} \tag{25}$$

and functions $Z_i : \{0,1,\cdots,s-1\} \to \mathbb{R},\ i = 1,\cdots, \omega\beta$, by

$$Z_i(l) = \bar{y}^{\mathrm{T}}(l+1)\ \bar{L}\ \delta_{\omega\beta}^i\ \bar{x}^*(l). \tag{26}$$

For any $l \in \{0,1,\cdots,s-1\}$, if for some integer i, $Z_i(l) > Z_j(l)$ for all $j \neq i$, we take $\sigma^*(l)\bar{u}^*(l) = \delta_{\omega\beta}^i$. Then $\sigma^*(l) = \delta_\omega^{i_1},\ \bar{u}^*(l) = \delta_\beta^{i_2}$, where $(i_1-1)\beta + i_2 = i$.

Proof Fix an any integer $l \in \{0,1,\cdots,s-1\}$ and an any vector $v \in \Delta_{\omega\beta}$. A new vector $\psi \in \Delta_{\omega\beta}$ is defined as:

$$\psi(j) = \begin{cases} v, & j = l, \\ \sigma^*(j)\bar{u}^*(j), & \text{otherwise}. \end{cases} \tag{27}$$

That is, ψ is equal to the product of the proper switching signal σ^* and the optimal control $\bar{u}^*$ except at the time l. Let $\bar{x}^*(t) = \bar{x}^*(t;\sigma,\bar{u})$ be the corresponding trajectory of the SBCN (18). By Eq.(20), we have

$$\begin{aligned} &\bar{x}^*(s) = \\ &\bar{L}\sigma^*(s-1)\bar{u}^*(s-1)\cdots\bar{L}\sigma^*(l)\bar{u}^*(l)\bar{x}^*(l) = \\ &H(s,l+1;\sigma^*,\bar{u}^*)\bar{L}\sigma^*(l)\bar{u}^*(l)\bar{x}^*(l). \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{x}(s) = &\bar{L}\sigma(s-1)\bar{u}(s-1)\cdots\bar{L}\sigma(l)\bar{u}(l)\bar{x}(l) = \\ &\bar{L}\sigma^*(s-1)\bar{u}^*(s-1)\cdots\bar{L}\sigma(l)\bar{u}(l)\bar{x}(l) = \\ &H(s,l+1;\sigma^*,\bar{u}^*)\bar{L}v\bar{x}^*(l). \end{aligned}$$

Thus,

$$\begin{aligned} &\bar{J}(\sigma^*,\bar{u}^*) - \bar{J}(\sigma,\bar{u}) = \\ &\bar{r}^{\mathrm{T}}(\bar{x}^*(s) - \bar{x}(s)) \\ &\bar{r}^{\mathrm{T}}H(s,l+1;\sigma^*,\bar{u}^*)\bar{L}(\sigma^*(l)\bar{u}^*(l) - v)\bar{x}^*(l) \\ &\widetilde{y}^{\mathrm{T}}(l+1)\bar{L}(\sigma^*(l)\bar{u}^*(l) - v)\bar{x}^*(l), \end{aligned}$$

where $\widetilde{y}^{\mathrm{T}}(l+1) = \bar{r}^{\mathrm{T}}H(s,l+1;\sigma^*,\bar{u}^*)$, then $\widetilde{y}^{\mathrm{T}}(s) = \bar{r}^{\mathrm{T}}$. According to (22), we obtain

$$H(s,l;\sigma^*,\bar{u}^*) = H(s,l+1;\sigma^*,\bar{u}^*)\bar{L}\sigma^*(l)\bar{u}^*(l).$$

Hence,

$$\begin{aligned} \widetilde{y}(l) = &(H(s,l;\sigma^*,\bar{u}^*))^{\mathrm{T}}\bar{r} = \\ &(L\sigma^*(l)\bar{u}^*(l))^T\widetilde{y}(l+1), \end{aligned}$$

which means that $\widetilde{y}(l) = y(l)$ for all $l \in \{0,1,\cdots,s-1\}$. Then

$$\begin{aligned} &\bar{J}(\sigma^*,\bar{u}^*) - \bar{J}(\sigma,\bar{u}) = \\ &y^{\mathrm{T}}(l+1)\bar{L}(\sigma^*(l)\bar{u}^*(l) - v)\bar{x}^*(l). \end{aligned} \tag{28}$$

Assume that there is an integer $i \in \{1,2,\cdots,\omega\beta\}$ satisfying $Z_i(l) > Z_j(l)$ for all $j = 1,2,\cdots,\omega\beta$ and $j \neq i$. We need to show that $\sigma^*(l)\bar{u}^*(l) = \delta_{\omega\beta}^i$.

Assuming $\sigma^*(l)\bar{u}^*(l) = \delta_{\omega\beta}^j,\ j \neq i$, and setting $v = \delta_{\omega\beta}^i$, (28) becomes

$$\begin{aligned} &\bar{J}(\sigma^*,\bar{u}^*) - \bar{J}(\sigma,\bar{u}) = \\ &y^{\mathrm{T}}(l+1)\bar{L}(\delta_{\omega\beta}^j - \delta_{\omega\beta}^i)\bar{x}^*(l) < 0. \end{aligned}$$

This contradicts the optimality of σ^* and $\bar{u}^*$. Hence, $\sigma^*(l)\bar{u}^*(l) = \delta_{\omega\beta}^i$. By Lemma 2, calculate i_1 and i_2 such that $\delta_\omega^{i_1} \ltimes \delta_\beta^{i_2} = \delta_{\omega\beta}^i$. Then, the proper switching signal is $\sigma^*(l) = \delta_\omega^{i_1}$ and the optimal control is $\bar{u}^*(l) = \delta_\beta^{i_2}$, where $(i_1-1)\beta + i_2 = i$. QED.

Remark 4 If there is no integer i satisfying $Z_i(l) > Z_j(l)$ for all $j \neq i$, Theorem 3 will not be able to achieve the proper switching signal $\sigma^*(l)$ and the optimal control $\bar{u}^*(l)$. In fact, assume that there exists a set $I = \{I_1, I_2,\cdots,I_h\} \subset$

$\{1,2,\cdots,\omega\beta\}$ satisfying $Z_{I_1}(l) = Z_{I_2}(l) = \cdots Z_{I_h}(l) > Z_j(l)$ for all $j \notin I$. Set

$$\psi^1(j) = \begin{cases} \delta_{\omega\beta}^{I_p}, & j = l; \\ \sigma^*(j)\bar{u}^*(j), & \text{otherwise}, \end{cases} \tag{29}$$

$$\psi^2(j) = \begin{cases} \delta_{\omega\beta}^{I_q}, & j = l; \\ \sigma^*(j)\bar{u}^*(j), & \text{otherwise}, \end{cases} \tag{30}$$

where $I_p,\ I_q \in I$, and $I_p \neq I_q$. Based on the proof of Theorem 3, we obtain that the values of the cost functional $\bar{J}(\cdot)$ in (29) and (30) are equal. Hence,

$$\sigma^*(l)\bar{u}^*(l) \in \{\delta_{\omega\beta}^{I_1},\cdots,\delta_{\omega\beta}^{I_h}\},\ l = 0\ \cdots\ s-1.$$

According to Lemma 2, we can obtain the corresponding switching signal $\sigma^*(l)$ and control $\bar{u}^*(l)$. Moreover, if σ^* is a proper switching signal and $\bar{u}^*$ is an optimal control, let

$$\zeta(j) = \begin{cases} v, & j = l; \\ \sigma^*(j)\bar{u}^*(j), & \text{otherwise}, \end{cases} \tag{31}$$

where $v \in \{\delta_{\omega\beta}^{I_1},\cdots,\delta_{\omega\beta}^{I_h}\}$, then ζ can also maximize the cost function $\bar{J}(\cdot)$.

According to the above analysis, we will provide an algorithm to design the proper switching sequence and control strategy such that the cost functional $\bar{J}(\cdot)$ in (24) is maximized at the fixed termination time s.

Algorithm 1

Step 1 Calculate the matrix $\bar{L}$ by (16)–(18).

Step 2 With $\bar{L}$, we can first gain all the functions $Z_i(s-1)(i = 1,\cdots,\omega\beta)$ in (26). If there exists an integer i_{s-1} satisfying $Z_{i_{s-1}}(s-1) \geqslant Z_j(s-1)$ for all $j \in \{1,2,\cdots,\omega\beta\}$, then $\sigma^*(s-1)\bar{u}^*(s-1) = \delta_{\omega\beta}^{i_{s-1}}$. Compute $i_{1_{s-1}}$ and $i_{2_{s-1}}$ by Lemma 2 such that $\delta_{\omega\beta}^{i_{s-1}} = \delta_{\omega}^{i_{1_{s-1}}}\delta_{\beta}^{i_{2_{s-1}}}$. Set $\sigma^*(s-1) = \delta_{\omega}^{i_{1_{s-1}}}$ and $\bar{u}^*(s-1) = \delta_{\beta}^{i_{2_{s-1}}}$.

Step 3 Calculate $y(s-1)$ by submitting $\sigma^*(s-1)\bar{u}^*(s-1)$ to (25). Set $l = s-2$.

Step 4 Calculate the functions $Z_i(l)$ in (26). If there exists an integer i_l satisfying $Z_{i_l}(l) \geqslant Z_j(l)$ for all j, then $\sigma^*(l)\bar{u}^*(l) = \delta_{\omega\beta}^{i_l}$. Compute i_{1_l} and i_{2_l} by Lemma 2 such that $\delta_{\omega\beta}^{i_l} = \delta_{\omega}^{i_{1_l}}\delta_{\beta}^{i_{2_l}}$. Set $\sigma^*(l) = \delta_{\omega}^{i_{1_l}}$ and $\bar{u}^*(l) = \delta_{\beta}^{i_{2_l}}$.

Step 5 If $l = 0$, Stop. Otherwise, calculate $y(l)$ in (25), set $l = l-1$, and return to Step 4.

Remark 5 It should be pointed out that Ref. [13] investigated a Mayer-type optimal control problem for Boolean control networks and derived a necessary condition for optimality via the Pontryagin maximum principle. Compared with Ref. [13], our main results have the following advantages: i) our results can be applied to the optimality analysis of switched singular Boolean control networks with both state and input constraints, while the results in Ref. [13] are only applicable to unconstrained Boolean control networks; ii) the computation complexity of Theorem 3 is $O(\alpha\beta)$, which is much less than the computation complexity of the results in [13], $O(2^{m+n})$, when $\alpha < 2^n$ and $\beta < 2^m$.

4 Illustrative example

In this section, we present an illustrative example to show how to use the results achieved in this paper to study the optimal control problem of SSBCNs with state and input constraints.

Example 1 Consider the following SSBCN:

$$\begin{cases} g_1^{\sigma(t)}(X(t+1)) = f_1^{\sigma(t)}(X(t),U(t)), \\ g_2^{\sigma(t)}(X(t+1)) = f_2^{\sigma(t)}(X(t),U(t)), \\ g_3^{\sigma(t)}(X(t+1)) = f_3^{\sigma(t)}(X(t),U(t)), \end{cases} \tag{32}$$

where $X(t) = (x_1(t),x_2(t),x_3(t)),\ U(t) = u(t)$, and

$$\begin{cases} g_1^1 = x_1 \vee [\neg x_1 \wedge (\neg x_2 \wedge \neg x_3)], \\ g_2^1 = x_1 \wedge (x_2 \vee x_3) \vee (\neg x_1 \wedge x_3), \\ g_3^1 = x_1 \wedge (x_2 \bar{\vee} x_3) \vee \neg x_1, \\ f_1^1 = u \wedge \{x_1 \vee [\neg x_1 \wedge (x_2 \vee \neg x_3)]\} \vee \\ \qquad \{\neg u \wedge \{[x_1 \wedge (x_2 \to \neg x_3)] \vee \\ \qquad [\neg x_1 \wedge (x_2 \to x_3)]\}\}, \\ f_2^1 = u \wedge \{[x_1 \wedge (\neg x_2 \wedge x_3)] \vee \\ \qquad [\neg x_1 \wedge (\neg x_2 \wedge \neg x_3)]\} \vee \\ \qquad \{\neg u \wedge \{(x_1 \wedge \neg x_2) \vee \\ \qquad [\neg x_1 \wedge (\neg x_2 \wedge \neg x_3)]\}\}, \\ f_3^1 = u \wedge \{[x_1 \wedge (\neg x_2 \wedge x_3)] \vee (\neg x_1 \wedge x_3)\} \vee \\ \qquad \{\neg u \wedge \{[x_1 \wedge (x_2 \wedge x_3)] \vee \\ \qquad [\neg x_1 \wedge (x_2 \bar{\vee} x_3)]\}\}, \\ g_1^2 = x_1 \wedge x_3 \vee [\neg x_1 \wedge (x_2 \vee x_3)], \\ g_2^2 = x_1 \wedge (\neg x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge \neg x_3), \\ g_3^2 = x_1 \wedge (x_2 \wedge x_3) \vee [\neg x_1 \wedge (\neg x_2 \wedge x_3)], \\ f_1^2 = u \wedge \{x_1 \wedge (x_2 \to x_3) \vee [\neg x_1 \wedge \\ \qquad (x_2 \vee \neg x_3)]\} \vee \{\neg u \wedge [x_1 \vee (\neg x_1 \wedge \neg x_3)]\}, \\ f_2^2 = u \wedge (x_1 \wedge x_2 \vee \neg x_1) \vee \{\neg u \wedge \\ \qquad \{[x_1 \wedge (x_2 \vee \neg x_3)] \vee [\neg x_1 \wedge (x_2 \vee x_3)]\}\}, \\ f_2^3 = u \wedge [x_1 \wedge (x_2 \to x_3) \vee (\neg x_1 \wedge x_2)] \vee \\ \qquad \{\neg u \wedge [x_1 \wedge (x_2 \bar{\vee} x_3) \vee (\neg x_1 \wedge \neg x_3)]\}. \end{cases}$$

Firstly, we convert the SSBCN (32) into an equivalent SBCN. Setting $x(t) = \ltimes_{i=1}^3 x_i(t)$, we have the following algebraic form:

$$E_{\sigma(t)}x(t+1) = F_{\sigma(t)}u(t)x(t), \tag{33}$$

where

$$E_1 = \delta_8[2\ 1\ 1\ 4\ 5\ 7\ 5\ 3],$$
$$E_2 = \delta_8[3\ 8\ 4\ 6\ 2\ 4\ 1\ 8],$$
$$F_1 = \delta_8[4\ 4\ 2\ 3\ 3\ 4\ 7\ 2\ 7\ 4\ 2\ 2\ 4\ 7\ 3\ 2],$$
$$F_2 = \delta_8[1\ 6\ 3\ 3\ 1\ 1\ 6\ 2\ 2\ 1\ 3\ 2\ 6\ 1\ 6\ 3].$$

Obviously, Lemma 3 holds for the SSBCN (33). According to Theorem 1, we obtain the equivalent SBCN for (33):

$$x(t+1) = L_{\sigma(t)}u(t)x(t), \tag{34}$$

where $L_1 = \delta_8[4\ 4\ 1\ 8\ 8\ 4\ 6\ 1\ 6\ 4\ 1\ 1\ 4\ 6\ 8\ 1]$ and $L_2 = \delta_8[7\ 4\ 1\ 1\ 7\ 7\ 4\ 5\ 5\ 7\ 1\ 5\ 4\ 7\ 4\ 1]$.

Secondly, consider the constrained SBCN (34). Assume that $C_{\mathrm{x}} = \{\delta_8^1, \delta_8^3, \delta_8^4, \delta_8^5, \delta_8^7, \delta_8^8\}$ and $C_{\mathrm{u}} = \{\delta_2^1, \delta_2^2\}$. By Theorem 2, the state trajectories of the SBCN (34) with C_{x} and C_{u} are equivalent to those of the following SBCN with $\bar{C}_{\mathrm{x}}$ and $\bar{C}_{\mathrm{u}}$:

$$\bar{x}(t+1) = \bar{L}\sigma(t)\bar{u}(t)\bar{x}(t), \tag{35}$$

where

$$\bar{L} = \delta_6[3\ 1\ 6\ 6\ 0\ 1\ 0\ 1\ 1\ 3\ 6\ 1\ 5\ 1\ 1\ 5\ 3\ 4\ 4\ 1\ 4\ 3\ 3\ 1] \in \mathcal{B}_{6\times 24}.$$

Finally, we consider the Mayer-type optimal control problem of the SSBCN (33) with C_{x} and C_{u}. The given vector $r^{\mathrm{T}} = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]$, and we aim to find the maximum value of the cost functional

$$J(\sigma, u; x(0)) = r^{\mathrm{T}}x(s) \tag{36}$$

under the initial value $x(0) = \delta_8^3$ and $s = 4$. This is equivalent to determining a proper switching sequence and a control strategy steering the initial value to $x_1(4) = 0,\ x_2(4) = x_3(4) = 1$, if they exist.

Based on Eq.(24), the cost functional $J(\cdot)$ in (36) becomes

$$\bar{J}(\sigma, \bar{u}; \bar{x}(0)) = \bar{r}^{\mathrm{T}}\bar{x}(s), \tag{37}$$

where $\bar{x}(0) = \delta_6^2$ and $\bar{r}^{\mathrm{T}} = [0\ 0\ 0\ 1\ 0\ 0]$.

Consider the functions

$$Z_i(3) = \bar{r}^{\mathrm{T}}\bar{L}\delta_{\omega\beta}^i\bar{x}^*(3) = [0\ 0\ 0\ 1\ 0\ 0]\bar{L}\delta_4^i\bar{x}^*(3),\ i = 1,2,3,4. \tag{38}$$

With simple calculation, we obtain $Z_1(3) = Z_2(3) = 0,\ Z_3(3) = [0\ 0\ 0\ 0\ 0\ 1]\bar{x}^*(3)$ and $Z_4(3) = [1\ 0\ 1\ 0\ 0\ 0]\bar{x}^*(3)$. Assuming $\bar{x}^*(3) = \delta_6^6$, then $Z_3(3) = 1$ and $Z_i(3) = 0$ for any $i \neq 3$. Based on Algorithm 1, we get $\sigma^*(3)\bar{u}^*(3) = \delta_4^3$, which means that $\sigma^*(3) = \delta_2^2$ and $\bar{u}^*(3) = \delta_2^1$.

Using (25) yields

$$y(3) = (\bar{L}\sigma^*(3)\bar{u}^*(3))^{\mathrm{T}}\bar{r} = (\bar{L}\delta_4^3)^{\mathrm{T}}[0\ 0\ 0\ 1\ 0\ 0]^{\mathrm{T}} = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]^{\mathrm{T}}, \tag{39}$$

thus

$$Z_i(2) = y^{\mathrm{T}}(3)\bar{L}\delta_{\omega\beta}^i\bar{x}^*(2) = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]\bar{L}\delta_4^i\bar{x}^*(2), \tag{40}$$

and this yields $Z_1(2) = [0\ 0\ 1\ 1\ 0\ 0]\bar{x}^*(2),\ Z_2(2) = [0\ 0\ 0\ 0\ 1\ 0]\bar{x}^*(2)$ and $Z_3(2) = Z_4(2) = 0$. Assume that $\bar{x}^*(2) = \delta_6^3$. Then $Z_1(2) > Z_j(2)$ for any $j \neq 1$, so Algorithm 1 means that $\sigma^*(2)\bar{u}^*(2) = \delta_4^1$. By Lemma 2, we have $\sigma^*(2) = \delta_2^1$ and $\bar{u}^*(2) = \delta_2^1$.

At this time,

$$y(2) = (\bar{L}\sigma^*(2)\bar{u}^*(2))^{\mathrm{T}}y(3) \\ (\bar{L}\delta_4^1)^{\mathrm{T}}[0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]^{\mathrm{T}} \\ [0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0]^{\mathrm{T}}, \tag{41}$$

and

$$Z_i(1) = y^{\mathrm{T}}(2)\bar{L}\delta_{\omega\beta}^i\bar{x}^*(1) = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]\bar{L}\delta_4^i\bar{x}^*(1),\ i = 1,2,3,4. \tag{42}$$

That is, $Z_1(1) = [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\bar{x}^*(1),\ Z_2(1) = [0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0]\bar{x}^*(1),\ Z_3(1) = [0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1]\bar{x}^*(1)$ and $Z_4(1) = [1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 0]\bar{x}^*(1)$. Assuming $\bar{x}^*(1) = \delta_6^1$, then $Z_1(1) = Z_4(1) = 1$ and $Z_2(1) = Z_3(1) = 0$. According to Remark 4 , we have $\sigma^*(1)\bar{u}^*(1) \in \{\delta_4^1, \delta_4^4\}$. Take $\sigma^*(1)\bar{u}^*(1) = \delta_4^1$, then $\sigma^*(1) = \delta_2^1$ and $\bar{u}^*(1) = \delta_2^1$. This yields

$$y(1) = (\bar{L}\sigma^*(1)\bar{u}^*(1))^{\mathrm{T}}y(2) = (\bar{L}\delta_4^1)^{\mathrm{T}}[0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0]^{\mathrm{T}} \\ [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^{\mathrm{T}}, \tag{43}$$

and

$$Z_i(0) = y^{\mathrm{T}}(1)\bar{L}\delta_{\omega\beta}^i\bar{x}^*(0) = [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\bar{L}\delta_4^i\delta_6^2,\ i = 1,2,3,4. \tag{44}$$

A simple calculation shows that $Z_i(0) = 1,\ i = 1,2,3,4$. Similarly, we obtain $\sigma^*(0)\bar{u}^*(0) \in \varDelta_4$. Take $\sigma^*(0)\bar{u}^*(0) = \delta_4^3$, then $\sigma^*(0) = \delta_2^2$ and $\bar{u}^*(0) = \delta_2^1$.

Summarizing, we obtain a proper switching sequence $\pi^* = \{(0, \delta_2^2), (1, \delta_2^1), (2, \delta_2^1), (3, \delta_2^2)\}$ and a control strategy $\{\bar{u}^*(0) = \delta_2^1,\ \bar{u}^*(1) = \delta_2^1,\ \bar{u}^*(2) = \delta_2^1, \bar{u}^*(3) = \delta_2^1\}$ that maximize the cost functional $\bar{J}(.)$ at time $s = 4$.

In fact, the proper switching sequence and control strategy are not unique. For example, when $\sigma^*(0)\bar{u}^*(0) \in \varDelta_4$, we can also take $\sigma^*(0)\bar{u}^*(0) = \delta_4^2$, which means the switching signal $\sigma^*(0) = \delta_2^1$ and the control $\bar{u}^*(0) = \delta_2^2$.

It is easy to get that the corresponding state trajectories of the SSBCN (33) are $x(0) = \delta_8^3,\ x(1) = \delta_8^1,\ x(2) = \delta_8^4,\ x(3) = \delta_8^8,\ x(4) = \delta_8^5$. Obviously, all $x(s) \in C_{\mathrm{x}},\ s = 0,1,2,3,4$. At this time, $J(\sigma^*, u^*; x(0)) = r^{\mathrm{T}}x(4) = 1$, so the proper switching sequence and control strategy obtained from the above analysis indeed force the SSBCN (33) to the desired location.

5 Conclusions

In this paper, using the STP-based framework, we have investigated the optimal control problem of SSBCNs with state and input constraints. Based on the algebraic form, a necessary and sufficient condition for the uniqueness of solution of the SSBCN has been discussed. The paralled constrained algebraic form of the constrained SSBCN has been obtained by converting a SSBCN into an equivalent SBCN. Then a necessary condition for the existence of optimal control has been proposed via a homologous needle variation. In addition, an algorithm has been given to design the proper switching sequence and control strategy that maximizes the cost functional at a fixed termination time.

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