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一类非Lipschitz非线性系统全局渐近稳定观测器设计

沈艳军[1†], 夏小华[2]

(1. 三峡大学 电气与新能源学院, 湖北 宜昌 443002;

2. 比勒陀利亚大学 电气工程与计算机学院, 南非 比勒陀利亚)

摘要: 本文讨论一类一致可测非线性系统全局渐近稳定观测器的设计. 所讨论的非线性系统是非Lipschitz的, 具有混合指数幂(指数幂大于1或小于1). 所设计的观测器有两项齐次误差项, 其中一项指数幂大于1, 另一项指数幂小于1. 此外, 所设计的观测器结构简单. 通过构造适当的齐次Lyapuonv函数证明了状态误差渐近收敛性. 最后两个数值仿真实验验证本方法的有效性.

关键词: 齐次; 渐近稳定; 非线性系统; 观测器

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Global asymptotic stable observers for a class of non-Lipschitz nonlinear systems

SHEN Yan-jun[1†], XIA Xiao-hua[2]

(1. College of Electrical Engineering and New Energy, China Three Gorges University, Yichang Hubei 443002, China;

2. Department of Electrical, Electronic and Computer Engineering, University of Pretoria, Pretoria, South Africa)

Abstract: A global asymptotic stable observer is proposed for a class of nonlinear systems with uniform observability. The characteristic of the systems is the non-Lipschitz conditions with mixed rational powers (greater than 1 or less than 1). The designed observers have two homogeneous terms of the observer error, one is with the power greater than 1 and the other is with the power smaller than 1. Moreover, the observers have explicit forms. By constructing a proper homogeneous Lyapunov function, we can obtain the asymptotic stability. Finally, two numerical simulations are given to show the validity of the proposed methods.

Key words: homogeneity; asymptotic stability; nonlinear system; observer

1 Introduction

For the last three decades, nonlinear state estimation has received a great deal of attraction in control theory. Many methods are proposed, for example, the Lyapunov based approach[1–2], moving horizon observers[3], the observer canonical form approach[4–6], high-gain observers[7–8]. Among these methods, high gain observers play an important role to estimate the unknown state of a relatively wide class of nonlinear systems. Recently, many works have investigated observer design in a lot of directions by extending the classes of nonlinear systems that admit global high gain observers[9]. There is a basic assumption: Lipschitz conditions, for the nonlinear systems with uniform observability form[9]. Recently, the authors in [8] made an attempt to relax this assumption by proposing observer design for systems with output dependent incremental rate.

There are further extensions to observer design for nonlinear systems with non-Lipschitz terms[10–12]. For instance, the bi-limit observer was presented to deal with these non-Lipschitz terms[13]. The observer design was done recursively in conjunction with an appropriate error Lyapunov function which is homogeneous in the bi-limit.

The main contribution of the paper is that, a homogeneous Lyapunov function is constructed and some fundamental inequalities are obtained for a class of homogeneous nonlinear systems. The inequalities and the homogeneous Lyapunov functions become tools for the design of asymptotically stable observers for more general class of nonlinear systems. The observers can also be explicitly constructed and the design parameters can be easily determined, in contrast to the recursive proce-

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†Corresponding author: E-mail: shenyj@ctgu.edu.cn.

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dures of [10,13–14]. Further, the systems considered in this paper include thus as a proper subset of the class of nonlinear systems considered in [10].

This paper is organized as follows. In Section 2, we present the main results: the global asymptotic stable observers for a class of nonlinear systems with non-Lipschitz terms are constructed. Two numerical examples are given to show the validity of our method in Section 3. Finally, the paper is concluded in Section 4.

2 Global asymptotic stable observers

In this section, our aim is to design a global asymptotic stable observer for the following system

$$\begin{cases} \dot{x}_1 = x_2 + f_1(y,u), \\ \dot{x}_2 = x_3 + f_2(y,x_2,u), \\ \quad \vdots \\ \dot{x}_n = f_n(y,x_2,\cdots,x_n,u), \\ y = x_1 = Cx,\ C = [1\ 0\ \cdots\ 0], \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the unknown state, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}$ is the measured output, $f_i(\cdot)$ $(i = 2,\cdots,n)$ are continuous functions and satisfy

$$\begin{aligned} &|f_i(y,x_2,\cdots,x_i,u) - f_i(y,\hat{x}_2,\cdots,\hat{x}_i,u)| \leqslant \\ &\sum_{j=2}^{i} l_{1,ij}|x_j - \hat{x}_j|^{p_{1,ij}} + l_{2,i}\sum_{j=2}^{i}|x_j - \hat{x}_j| + \\ &\sum_{j=2}^{i} l_{3,ij}|x_j - \hat{x}_j|^{p_{2,ij}}, \end{aligned} \tag{2}$$

where $\dfrac{n-i}{n-j+1} < p_{1,ij} < 1$, $1 < p_{2,ij} < \dfrac{i}{j-1}$, $l_{1,ij}, l_{2,i}, l_{3,ij} \geqslant 0$ and $\sum\limits_{i=2}^{n}(\sum\limits_{j=2}^{i}(l_{1,ij}^2 + l_{3,ij}^2) + l_{2,i}^2) > 0$.

The following remark is given to better understand the condition (2).

Remark 1 Consider the following system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3 - x_1 - x_2, \\ \dot{x}_3 = x_4 - x_2, \\ \dot{x}_4 = f_4(x_1,x_2,x_3,x_4), \\ y = x_1, \end{cases} \tag{3}$$

where

$$f_4(x_1,x_2,x_3,x_4) = -x_3 + \frac{x_4x_2^2}{(1+x_4^2)(1+x_2^{2/3})} + x_4^{3/5} - x_4.$$

It can be verified

$$\begin{aligned} &|f_4(x_1,x_2,x_3,x_4) - f_4(x_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)| < \\ &16|x_2 - \hat{x}_2| + 12|x_2 - \hat{x}_2|^2 + |x_3 - \hat{x}_3| + \\ &2|x_4 - \hat{x}_4|^{3/5} + |x_4 - \hat{x}_4|. \end{aligned}$$

Now, we investigate the uniqueness of solutions to the system (3). Construct the following Lyapunov function

$$V_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

The derivative of $V_1(x)$ along the solution to (3) is given as

$$\begin{aligned} &\frac{\mathrm{d}V_1(x)}{\mathrm{d}t}|_{(3)} = \\ &2x_4^{8/5} - 2x_4^2 + \frac{2x_4^2x_2^2}{(1+x_4^2)(1+x_2^{2/3})} - 2x_2^2 \leqslant \\ &2x_4^{8/5} - 2x_4^2. \end{aligned}$$

It is easy to obtain that when $|x_4| > 1$, $\dfrac{\mathrm{d}V_1(x)}{\mathrm{d}t}|_{(3)} < 0$. Therefore, (3) is locally Lipschitz everywhere except on $x_4 = 0$. Hence, forward uniqueness of solutions for all initial conditions except the origin follows from Proposition 8.1 in [15].

It is beyond the scope of this paper to discuss the general conditions for the existence and uniqueness of solutions to (1) and its associated observers to be designed. As in [10], we restrict our attention to estimating the solutions to systems that exist globally in positive time. More results on existence and uniqueness of possibly non-Lipschitz systems could be found in [16].

A global asymptotic stable observer for the system (1) can be designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \theta[S]_{11}^{-1}\lceil e_1\rfloor^{\alpha_1} + \theta[S]_{11}^{-1}\lceil e_1\rfloor^{\beta_1} + \\ \qquad f_1(y,u), \\ \dot{\hat{x}}_2 = \hat{x}_3 + \theta[S]_{21}^{-1}\lceil e_1\rfloor^{\alpha_2} + \theta[S]_{21}^{-1}\lceil e_1\rfloor^{\beta_2} + \\ \qquad f_2(y,\hat{x}_2,u), \\ \quad \vdots \\ \dot{\hat{x}}_n = \theta[S]_{n1}^{-1}\lceil e_1\rfloor^{\alpha_n} + \theta[S]_{n1}^{-1}\lceil e_1\rfloor^{\beta_n} + \\ \qquad f_n(y,\hat{x}_2,\cdots,\hat{x}_n,u), \end{cases} \tag{4}$$

where $e_1 = x_1 - \hat{x}_1$, $\lceil e_1\rfloor^{\alpha_i} = |e_1|^{\alpha_i}\mathrm{sgn}\, e_1$, α_i and β_i are given by

$$\alpha_i = i\alpha - (i-1),\ \beta_i = i\beta - (i-1), \tag{5}$$

and $1 - \dfrac{1}{n} < \alpha < 1$, $\beta > 1$, and $[S]_{i1}^{-1}$ denotes element of n-dimensional symmetric matrix $S(\theta)^{-1}$ at the ith row and the first column with $\theta = 1$, and $S(\theta)$ satisfies

$$-\theta S(\theta) - A\mathrm{T}_0 S(\theta) - S(\theta)A_0 + C^{\mathrm{T}}C = 0,\ S(\theta) > \underline{s}I, \tag{6}$$

where $A_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, $\theta \geqslant 1$, $\underline{s} > 0$ are two constants[7]. The dynamics of $e = x - \hat{x}$ is given by

$$\begin{cases} \dot{e}_1 = e_2 - \theta[S]_{11}^{-1}\lceil e_1\rfloor^{\alpha_1} - \theta[S]_{11}^{-1}\lceil e_1\rfloor^{\beta_1}, \\ \dot{e}_2 = e_3 - \theta[S]_{21}^{-1}\lceil e_1\rfloor^{\alpha_2} - \theta[S]_{21}^{-1}\lceil e_1\rfloor^{\beta_2} + \tilde{f}_2, \\ \quad \vdots \\ \dot{e}_n = -\theta[S]_{n1}^{-1}\lceil e_1\rfloor^{\alpha_n} - \theta[S]_{n1}^{-1}\lceil e_1\rfloor^{\beta_n} + \tilde{f}_n, \end{cases}$$

where

$$\tilde{f}_i = f_i(y,x_2,\cdots,x_i,u) - f_i(y,\hat{x}_2,\cdots,\hat{x}_i,u).$$

Consider the change of coordinates $\varepsilon_i = \dfrac{e_i}{\theta^{i-1+\sigma}}$, where $0 < \sigma < 1$. Then,

$$\begin{cases}\dot{\varepsilon}_1=\theta\varepsilon_2-\theta^{(\alpha_1-1)\sigma+1}[S]_{11}^{-1}\lceil\varepsilon_1\rfloor^{\alpha_1}-\\ \qquad\theta^{(\beta_1-1)\sigma+1}[S]_{11}^{-1}\lceil\varepsilon_1\rfloor^{\beta_1},\\ \dot{\varepsilon}_2=\theta\varepsilon_3-\theta^{(\alpha_2-1)\sigma}[S]_{21}^{-1}\lceil\varepsilon_1\rfloor^{\alpha_2}-\\ \qquad\theta^{(\beta_2-1)\sigma}[S]_{21}^{-1}\lceil\varepsilon_1\rfloor^{\beta_2}+\dfrac{\tilde{f}_2}{\theta^{1+\sigma}},\\ \quad\vdots\\ \dot{\varepsilon}_n=-\theta^{(\alpha_n-1)\sigma-n+2}[S]_{n1}^{-1}\lceil\varepsilon_1\rfloor^{\alpha_n}-\\ \qquad\theta^{(\beta_n-1)\sigma-n+2}[S]_{n1}^{-1}\lceil\varepsilon_1\rfloor^{\beta_n}+\dfrac{\tilde{f}_n}{\theta^{n-1+\sigma}}.\end{cases}\tag{7}$$

Now, we can give the following result.

Theorem 1 If $\dfrac{n-i}{n-j+1}<p_{1,ij}<1,\ 1<p_{2,ij}<\dfrac{i}{j-1}(2\leqslant j\leqslant i\leqslant n)$, for any α, $1-\dfrac{1}{n}<\alpha<1$, any σ, $0<\sigma<\min\{\dfrac{i-p_{2,ij}(j-1)}{1+p_{2,ij}},\dfrac{i-p_{2,ij}(j-1)}{2(p_{2,ij}-1)},1\}$, any β, $1+\dfrac{1}{2\sigma}<\beta<1+\dfrac{1}{\sigma}$, there exists $\theta\geqslant 1$, such that (4) is a global asymptotic observer of the system (1) with the condition (2).

In order to prove Theorem 1, we need extra properties on the following system:

$$\begin{cases}\dot{\varepsilon}_1=\theta\varepsilon_2-\theta^{(\lambda_1-1)\sigma+1}[S]_{11}^{-1}\lceil\varepsilon_1\rfloor^{\lambda_1},\\ \quad\vdots\\ \dot{\varepsilon}_{n-1}=\theta\varepsilon_n-\theta^{(\lambda_{n-1}-1)\sigma-n+3}[S]_{n-1,1}^{-1}\lceil\varepsilon_1\rfloor^{\lambda_{n-1}},\\ \dot{\varepsilon}_n=-\theta^{(\lambda_n-1)\sigma-n+2}[S]_{n1}^{-1}\lceil\varepsilon_1\rfloor^{\lambda_n},\end{cases}\tag{8}$$

where $\lambda>0$ is a constant, $\lambda_i=i\lambda-(i-1),\ i=0,1,\cdots,n$. We have following results for the system (8).

Proposition 1 Let $\kappa(s)\in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\kappa(s)=\begin{cases}0, \text{ on } (-\infty,1],\\ 1, \text{ on } [2,\infty),\end{cases}$ and its derivative satisfying $\kappa'(s)\geqslant 0$ on $(-\infty,\infty)$ and $0<\underline{\kappa}\leqslant\kappa'(s)\leqslant\bar{\kappa}$ on $[\dfrac{5}{4},\dfrac{7}{4}]$ for two bounds $\underline{\kappa}$ and $\bar{\kappa}$. Construct the following function as in[17]

$$V_\theta^\lambda(\varepsilon)=\begin{cases}\displaystyle\int_0^\infty\frac{1}{v^{r_1+1}}(\kappa\circ\bar{V})(v\varepsilon_1,\cdots,v^{\lambda_{n-1}}\varepsilon_n)\mathrm{d}v,\\ \qquad\qquad\qquad\qquad\varepsilon\in\mathbb{R}^n\backslash\{0\},\\ 0,\qquad\qquad\qquad\quad\ \varepsilon=0,\end{cases}\tag{9}$$

where r_1 is a positive integer and $\bar{V}(\varepsilon)=\varepsilon^{\mathrm{T}}S\varepsilon$, where $S=S(1)$. Then, we have

i) $V_\theta^\lambda(\varepsilon)$ is a positive definite function with homogeneity of degree r_1 w.r.t. the weights $\{\lambda_{i-1}\}_{1\leqslant i\leqslant n}$. For convenience, $V_\theta^\lambda(\varepsilon)$ is called as an r_1 h–Lyapunov function of $\bar{V}(\varepsilon)$ w.r.t. $\kappa,\theta,(\lambda_0,\lambda_1,\cdots,\lambda_{n-1})$.

ii) If $r_1>1+\max\{\lambda_i\}_{0\leqslant i\leqslant n-1}$ and $1-\dfrac{1}{n}<\lambda<1$, then, $V_\theta^\lambda(\varepsilon)$ is C^1 on $\mathbb{R}^n$, and

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}\leqslant-c_2\theta^{1-\sigma}V_\theta^\lambda(\varepsilon)^{\gamma_1},\ \varepsilon\in\mathbb{R}^n\backslash\{0\},\tag{10}$$

where $c_2>0,\ \gamma_1=\dfrac{r_1+\lambda-1}{r_1}$.

iii) If $r_1>1+\max\{\lambda_i\}_{0\leqslant i\leqslant n-1}$ and $1\leqslant\lambda<1+\dfrac{1}{\sigma}$, then, $V_\theta^\lambda(\varepsilon)$ is C^1 on $\mathbb{R}^n$, and

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}\leqslant-c_2'\theta V_\theta^\lambda(\varepsilon)^{\gamma_1},\ \varepsilon\in\mathbb{R}^n\backslash\{0\},\tag{11}$$

where $c_2'>0$, $\gamma_1=\dfrac{r_1+\lambda-1}{r_1}$.

Proof The proof of this Proposition is given in the Appendix.

The dedicated construction of homogeneous Lyapunov functions (9) and the fundamental inequality (10) become tools for the design of asymptotically stable observers for the system (1).

Proof of Theorem 1 Let $\mathcal{B}_{V_\theta^\beta,\delta}\triangleq\{\varepsilon:V_\theta^\beta(\varepsilon)\leqslant\delta\}$, where $V_\theta^\beta(\varepsilon)$ is $r_2(>\beta_n)$ h–Lyapunov function of $\bar{V}(\varepsilon)$. The proof is split into three parts. We will properly choose δ_2 and δ_4 in such a way that $\mathcal{B}_{V_\theta^\beta,\delta_4}\subset\mathcal{B}_{V_\theta^\beta,\delta_2}\subset\mathcal{B}_{V_\theta^\beta,1}$ as shown in Fig.1. In Part 1 and Part 2, we use an r_2 h–Lyapunov function V_θ^β to prove $\dot{V}_\theta^\beta$ is negative definite on $\mathbb{R}^n\backslash\mathcal{B}_{V_\theta^\beta,1}$ and $\mathcal{B}_{V_\theta^\beta,1}\backslash\mathcal{B}_{V_\theta^\beta,\delta_2}$, respectively. Then, an $r_3(>1)$ h–Lyapunov function V_θ^α of $\bar{V}(\varepsilon)$ is used to prove $\dot{V}_\theta^\alpha<0$ on $\mathcal{B}_{V_\theta^\beta,\delta_4}$ in Part 3. Lastly, we shall prove that $\mathcal{B}_{V_\theta^\beta,\delta_2}$ and $\mathcal{B}_{V_\theta^\beta,\delta_4}$ can be chosen arbitrarily close to each other. Then, since $\dot{V}_\theta^\alpha$ is continuous, we can obtain that $\dot{V}_\theta^\alpha<0$ on $\mathcal{B}_{V_\theta^\beta,\delta_2}\backslash\mathcal{B}_{V_\theta^\beta,\delta_4}$.

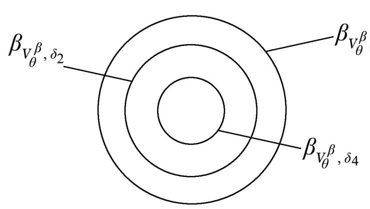

Fig. 1 The illustrative diagram of $\mathcal{B}_{V_\theta^\beta,\delta_1},\mathcal{B}_{V_\theta^\beta,\delta_2},\mathcal{B}_{V_\theta^\beta,\delta_4}$

Part 1 Note that

$$p_{2,ij}<\min\{\frac{i-\sigma}{j-1+\sigma},\frac{i+2\sigma}{j-1+2\sigma}\},\ 2\leqslant j\leqslant i\leqslant n,$$

and $1+\dfrac{1}{2\sigma}<\beta<1+\dfrac{1}{\sigma}$. Then, $\dfrac{\beta_i}{\beta_{j-1}}>\dfrac{i+2\sigma}{j-1+2\sigma}>p_{2,ij}$. Therefore, $p_{2,ij}\beta_{j-1}-\beta_{i-1}<\beta-1$. Calculate the derivative of $V_\theta^\beta(\varepsilon)$ along the solution to (7). By Proposition 1, we have

$$\frac{\mathrm{d}V_\theta^\beta(\varepsilon)}{\mathrm{d}t}|_{(7)}<-c_2'\theta V_\theta^\beta(\varepsilon)^{\gamma_2}-\frac{\partial V_\theta^\beta}{\partial\varepsilon}^{\mathrm{T}}\tilde{F}_1+\frac{\partial V_\theta^\beta}{\partial\varepsilon}^{\mathrm{T}}\tilde{F}_2,\tag{12}$$

where

$$\tilde{F}_1 = \begin{bmatrix} \theta^{(\alpha_1-1)\sigma+1}[S]_{11}^{-1}\lceil\varepsilon_1\rfloor^{\alpha_1} \\ \theta^{(\alpha_2-1)\sigma}[S]_{21}^{-1}\lceil\varepsilon_1\rfloor^{\alpha_2} \\ \vdots \\ \theta^{(\alpha_n-1)\sigma-n+2}[S]_{n1}^{-1}\lceil\varepsilon_1\rfloor^{\alpha_n} \end{bmatrix},$$

$$\tilde{F}_2 = \begin{bmatrix} 0 \\ \dfrac{\tilde{f}_2}{\theta^{1+\sigma}} \\ \vdots \\ \dfrac{\tilde{f}_n}{\theta^{n-1+\sigma}} \end{bmatrix},\ \gamma_2 = \frac{r_2+\beta-1}{r_2}.$$

When $\varepsilon \in \mathbb{R}^n \backslash \mathcal{B}_{V_\theta^\beta,1}$, note that $\dfrac{\partial V_\theta^\beta}{\partial \varepsilon_i}\lceil\varepsilon_1\rfloor^{\alpha_i}$, $\dfrac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j$, $\dfrac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j^{p_{1,ij}}$, $\dfrac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j^{p_{2,ij}}$ are homogeneous of degree $r_2-\beta_{i-1}+\alpha_i$, $r_2-\beta_{i-1}+\beta_{j-1}$, $r_2-\beta_{i-1}+p_{1,ij}\beta_{j-1}$ and $r_2-\beta_{i-1}+p_{2,ij}\beta_{j-1}$, respectively. Then, by Lemma 4.2 in [15], there exist c_3, c_4, c_5 and $c_6 > 0$ such that

$$\begin{aligned}
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\lceil\varepsilon_1\rfloor^{\alpha_i}| < c_3 V_\theta^\beta(\varepsilon)^{\frac{r_2-\beta_{i-1}+\alpha_i}{r_2}} < c_3 V_\theta^\beta(\varepsilon),\\
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j^{p_{1,ij}}| < c_4 V_\theta^\beta(\varepsilon)^{\frac{r_2-\beta_{i-1}+p_{1,ij}\beta_{j-1}}{r_2}} < \\
&\qquad c_4 V_\theta^\beta(\varepsilon)^{1+\frac{\bar{\beta}_1}{r_2}},\\
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j| < c_5 V_\theta^\beta(\varepsilon)^{\frac{r_2-\beta_{i-1}+\beta_{j-1}}{r_2}} < c_5 V_\theta^\beta(\varepsilon),\\
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j^{p_{2,ij}}| < c_6 V_\theta^\beta(\varepsilon)^{\frac{r_2-\beta_{i-1}+p_{2,ij}\beta_{j-1}}{r_2}} < \\
&\qquad c_6 V_\theta^\beta(\varepsilon)^{1+\frac{\bar{\beta}_2}{r_2}},
\end{aligned}$$

where

$$\bar{\beta}_1 = \max\{p_{1,ij}\beta_{j-1}-\beta_{i-1}\},\quad \bar{\beta}_2 = \max\{p_{2,ij}\beta_{j-1}-\beta_{i-1}\}.$$

Let $\bar{S} = \max\{|[S]_{i1}^{-1}|\}$. Then,

$$|\frac{\partial V_\theta^\beta}{\partial \varepsilon}^{\mathrm{T}}\tilde{F}_1| < \theta^{(\alpha_1-1)\sigma+1} n\bar{S}c_3 V_\theta^\beta(\varepsilon), \tag{13}$$

$$\begin{aligned}
&|\frac{\partial V_\theta^\beta(\varepsilon)}{\partial \varepsilon}^{\mathrm{T}}\tilde{F}_2| \leqslant\\
&\sum_{i=2}^{n}\sum_{j=2}^{i} l_{1,ij}|\frac{\partial V_\theta^\beta(\varepsilon)}{\partial \varepsilon_i}|\varepsilon_j|^{p_{1,ij}}| +\\
&\sum_{i=2}^{n} l_{2,i}\sum_{j=2}^{i}|\frac{\partial V_\theta^\beta(\varepsilon)}{\partial \varepsilon_i}|\varepsilon_j|| +\\
&\theta^{1-2\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i} l_{3,ij}|\frac{\partial V_\theta^\beta(\varepsilon)}{\partial \varepsilon_i}|\varepsilon_j|^{p_{2,ij}}| \leqslant\\
&l_1c_4n^2V_\theta^\beta(\varepsilon)^{1+\frac{\bar{\beta}_1}{r_2}} + l_2c_5n^2V_\theta^\beta(\varepsilon) +\\
&\theta^{1-2\sigma}l_3c_6n^2V_\theta^\beta(\varepsilon)^{1+\frac{\bar{\beta}_2}{r_2}},
\end{aligned} \tag{14}$$

where $l_1 = \max\{l_{1,ij}\}$, $l_2 = \max\{l_{2,i}\}$, $l_3 = \max\{l_{3,ij}\}$. It follows from (12)−(14) that

$$\begin{aligned}
&\frac{\mathrm{d}V_\theta^\beta(\varepsilon)}{\mathrm{d}t}|_{(7)} <\\
&-c_2'\theta V_\theta^\beta(\varepsilon)^{\gamma_2} + \theta^{(\alpha_1-1)\sigma+1}n\bar{S}c_3V_\theta^\beta(\varepsilon) +\\
&l_1c_4n^2V_\theta^\beta(\varepsilon)^{1+\frac{\bar{\beta}_1}{r_2}} + l_2c_5n^2V_\theta^\beta(\varepsilon) +\\
&\theta^{1-2\sigma}l_3c_6n^2V_\theta^\beta(\varepsilon)^{1+\frac{\bar{\beta}_2}{r_2}}.
\end{aligned} \tag{15}$$

There exists $\theta_1 > 0$ such that when $\theta > \theta_1$, we have

$$\begin{aligned}
&\theta^{(1-\alpha_1)\sigma} > \frac{4n\bar{S}c_3}{c_2'},\ \theta > \frac{4l_1c_4n^2}{c_2'},\\
&\theta > \frac{4l_2c_5n^2}{c_2'},\ \theta^{2\sigma} > \frac{4l_3c_6n^2}{c_2'}.
\end{aligned} \tag{16}$$

Then,

$$\frac{\mathrm{d}V_\theta^\beta(\varepsilon)}{\mathrm{d}t}|_{(7)} < 0,\ \varepsilon \in \mathbb{R}^n \backslash \mathcal{B}_{V_\theta^\beta,1}.$$

Part 2 When $\varepsilon \in \mathcal{B}_{V_\theta^\beta,1}$, for the same choices of c_3, c_4, c_5 and c_6, we have

$$\begin{aligned}
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\lceil\varepsilon_1\rfloor^{\alpha_i}| < c_3V_\theta^\beta(\varepsilon)^{1-\frac{\beta_{n-1}-\alpha_n}{r_2}},\\
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j^{p_{1,ij}}| < c_4V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_1}{r_2}},\\
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j| < c_5V_\theta^\beta(\varepsilon)^{1+\frac{\beta_1-\beta_{n-1}}{r_2}},\\
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon_i}\varepsilon_j^{p_{2,ij}}| < c_6V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_2}{r_2}},
\end{aligned}$$

where

$$\underline{\beta}_1 = \min\{p_{1,ij}\beta_{j-1}-\beta_{i-1}\} < 0,\quad \underline{\beta}_2 = \min\{p_{2,ij}\beta_{j-1}-\beta_{i-1}\} < \beta-1.$$

Therefore,

$$\begin{aligned}
&|\frac{\partial V_\theta^\beta}{\partial \varepsilon}^{\mathrm{T}}\tilde{F}_1| < \theta^{(\alpha_1-1)\sigma+1}n\bar{S}c_3V_\theta^\beta(\varepsilon)^{1-\frac{\beta_{n-1}-\alpha_n}{r_2}},\\
&|\frac{\partial V_\theta^\beta(\varepsilon)}{\partial \varepsilon}^{\mathrm{T}}\tilde{F}_2| \leqslant\\
&l_1c_4n^2V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_1}{r_2}} + l_2c_5n^2V_\theta^\beta(\varepsilon)^{1+\frac{\beta_1-\beta_{n-1}}{r_2}} +\\
&\theta^{1-2\sigma}l_3c_6n^2V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_2}{r_2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\mathrm{d}V_\theta^\beta(\varepsilon)}{\mathrm{d}t}|_{(7)} <\\
&-c_2'\theta V_\theta^\beta(\varepsilon)^{\gamma_2} + \theta^{(\alpha_1-1)\sigma+1}n\bar{S}c_3V_\theta^\beta(\varepsilon)^{1-\frac{\beta_{n-1}-\alpha_n}{r_2}} +\\
&l_1c_4n^2V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_1}{r_2}} + l_2c_5n^2V_\theta^\beta(\varepsilon)^{1+\frac{\beta_1-\beta_{n-1}}{r_2}} +\\
&\theta^{1-2\sigma}l_3c_6n^2V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_2}{r_2}}.
\end{aligned}$$

From

$$\begin{aligned}
&c_2'\theta V_\theta^\beta(\varepsilon)^{\gamma_2} > 4\theta^{(\alpha_1-1)\sigma+1}n\bar{S}c_3V_\theta^\beta(\varepsilon)^{1-\frac{\beta_{n-1}-\alpha_n}{r_2}},\\
&c_2'\theta V_\theta^\beta(\varepsilon)^{\gamma_2} > 4l_1c_4n^2V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_1}{r_2}},\\
&c_2'\theta V_\theta^\beta(\varepsilon)^{\gamma_2} > 4l_2c_5n^2V_\theta^\beta(\varepsilon)^{1+\frac{\beta_1-\beta_{n-1}}{r_2}},\\
&c_2'\theta^{2\sigma}V_\theta^\beta(\varepsilon)^{\gamma_2} > 4l_3c_6n^2V_\theta^\beta(\varepsilon)^{1+\frac{\underline{\beta}_2}{r_2}},
\end{aligned}$$

we have

$$V_\theta^\beta(\varepsilon) > (\frac{4n\bar{S}c_3}{c_2'})^{\frac{r_2}{\beta_n-\alpha_n}}\theta^{-\frac{(1-\alpha_1)\sigma r_2}{n(\beta-\alpha)}},$$

$$V_\theta^\beta(\varepsilon) > (\frac{4l_1c_4n^2}{c_2'})^{\frac{r_2}{\beta-1-\underline{\beta}_1}}\theta^{-\frac{r_2}{\beta-1-\underline{\beta}_1}},$$

$$V_\theta^\beta(\varepsilon) > (\frac{4l_2c_5n^2}{c_2'})^{\frac{r_2}{(n-1)(\beta-1)}}\theta^{-\frac{r_2}{(n-1)(\beta-1)}},$$

$$V_\theta^\beta(\varepsilon) > (\frac{4l_3c_6n^2}{c_2'})^{\frac{r_2}{\beta-1-\underline{\beta}_2}}\theta^{-\frac{2\sigma r_2}{\beta-1-\underline{\beta}_2}}.$$

Let

$$\delta_2 = \max\{(\frac{4l_1c_4n^2}{c_2'})^{\frac{r_2}{\beta-1-\underline{\beta}_1}}\theta^{-\frac{r_2}{\beta-1-\underline{\beta}_1}}, (\frac{4n\bar{S}c_3}{c_2'})^{\frac{r_2}{\beta_n-\alpha_n}}\theta^{-\frac{(1-\alpha_1)\sigma r_2}{n(\beta-\alpha)}}, (\frac{4l_3c_6n^2}{c_2'})^{\frac{r_2}{\beta-1-\underline{\beta}_2}}\theta^{-\frac{2\sigma r_2}{\beta-1-\underline{\beta}_2}}, (\frac{4l_2c_5n^2}{c_2'})^{\frac{r_2}{(n-1)(\beta-1)}}\theta^{-\frac{r_2}{(n-1)(\beta-1)}}\}.$$

Then, there exists $\theta_2 > 0$ such that $\theta > \theta_2$ and $\delta_2 < 1$. Therefore,

$$\frac{\mathrm{d}V_\theta^\beta(\varepsilon)}{\mathrm{d}t}|_{(7)} < 0,\ \varepsilon \in \mathcal{B}_{V_\theta^\beta,1}\backslash\mathcal{B}_{V_\theta^\beta,\delta_2}.$$

Part 3 Consider the $r_3(>1)$ h–Lyapunov function $V_\theta^\alpha(\varepsilon)$ of $\bar{V}(\varepsilon)$. When $\varepsilon \in \mathcal{B}_{V_\theta^\alpha,1}$, from Proposition 1, we have

$$\frac{\mathrm{d}V_\theta^\alpha(\varepsilon)}{\mathrm{d}t}|_{(7)} < -c_2\theta^{1-\sigma}V_\theta^\alpha(\varepsilon)^{\gamma_3} - \frac{\partial V_\theta^\alpha(\varepsilon)}{\partial\varepsilon}^{\mathrm{T}}\tilde{F}_3 + \frac{\partial V_\theta^\alpha(\varepsilon)}{\partial\varepsilon}^{\mathrm{T}}\tilde{F}_2, \tag{17}$$

$$\tilde{F}_3 = \begin{bmatrix} \theta^{(\beta_1-1)\sigma+1}[S]_{11}^{-1}\lceil\varepsilon_1\rfloor^{\beta_1} \\ \theta^{(\beta_2-1)\sigma}[S]_{21}^{-1}\lceil\varepsilon_1\rfloor^{\beta_2} \\ \vdots \\ \theta^{(\beta_n-1)\sigma-n+2}[S]_{n1}^{-1}\lceil\varepsilon_1\rfloor^{\beta_n} \end{bmatrix},\ \gamma_3 = \frac{r_3+\alpha-1}{r_3}.$$

By using the same method as the first part, there exist $c_7, c_8, c_9, c_{10} > 0$ such that

$$|\frac{\partial V_\theta^\alpha}{\partial\varepsilon_i}\lceil\varepsilon_1\rfloor^{\beta_i}| < c_7V_\theta^\alpha(\varepsilon)^{1+\frac{\beta-1}{r_3}},$$

$$|\frac{\partial V_\theta^\alpha}{\partial\varepsilon_i}\varepsilon_j^{p_{1,ij}}| < c_8V_\theta^\alpha(\varepsilon)^{1+\frac{\underline{\beta}_3}{r_3}},$$

$$|\frac{\partial V_\theta^\alpha}{\partial\varepsilon_i}\varepsilon_j| < c_9V_\theta^\alpha(\varepsilon),$$

$$|\frac{\partial V_\theta^\alpha}{\partial\varepsilon_i}\varepsilon_j^{p_{2,ij}}| < c_{10}V_\theta^\alpha(\varepsilon)^{1+\frac{\underline{\beta}_4}{r_3}},$$

where $\underline{\beta}_3 = \min\{-\alpha_{i-1} + p_{1,ij}\alpha_{j-1}\} > \alpha - 1$, $\underline{\beta}_4 = \min\{-\alpha_{i-1} + p_{2,ij}\alpha_{j-1}\} > \alpha - 1$. Then,

$$|\frac{\partial V_\theta^\alpha(\varepsilon)}{\partial\varepsilon}^{\mathrm{T}}\tilde{F}_3| < \theta^2\bar{S}c_7V_\theta^\alpha(\varepsilon)^{1+\frac{\beta-1}{r_3}},$$

$$|\frac{\partial V_\theta^\alpha(\varepsilon)}{\partial\varepsilon}^{\mathrm{T}}\tilde{F}_2| < l_1c_8n^2V_\theta^\alpha(\varepsilon)^{1+\frac{\underline{\beta}_3}{r_3}} + l_2c_9n^2V_\theta^\alpha(\varepsilon) + \theta^{1-2\sigma}l_3c_{10}n^2V_\theta^\alpha(\varepsilon)^{1+\frac{\underline{\beta}_4}{r_3}},$$

which yields

$$\frac{\mathrm{d}V_\theta^\alpha(\varepsilon)}{\mathrm{d}t}|_{(7)} < -c_2\theta^{1-\sigma}V_\theta^\alpha(\varepsilon)^{\gamma_3}\theta^2\bar{S}c_7V_\theta^\alpha(\varepsilon)^{1+\frac{\beta-1}{r_3}} + l_1c_8n^2V_\theta^\alpha(\varepsilon)^{1+\frac{\underline{\beta}_3}{r_3}} + l_2c_9n^2V_\theta^\alpha(\varepsilon) + \theta^{1-2\sigma}l_3c_{10}n^2V_\theta^\alpha(\varepsilon)^{1+\frac{\underline{\beta}_4}{r_3}}.$$

Let

$$\delta_3 = \min\{(\frac{c_2}{4\bar{S}c_7})^{\frac{r_3}{\beta-\alpha}}\theta^{-\frac{(1+\sigma)r_3}{\beta-\alpha}}, (\frac{c_2}{4l_1n^2c_8})^{\frac{r_3}{\underline{\beta}_3-\alpha+1}}\theta^{-\frac{(1-\sigma)r_3}{\underline{\beta}_3-\alpha+1}}, (\frac{c_2}{4l_3n^2c_{10}})^{\frac{r_3}{\underline{\beta}_4-\alpha+1}}\theta^{-\frac{\sigma r_3}{\underline{\beta}_4-\alpha+1}}, (\frac{c_2}{4l_2n^2c_9})^{\frac{r_3}{1-\alpha}}\theta^{-r_3(1-\sigma)}\}.$$

Then, there exists $\theta_3 > 0$ such that $\theta > \theta_3$ and $\delta_3 < 1$. Thus,

$$\frac{\mathrm{d}V_\theta^\alpha(\varepsilon)}{\mathrm{d}t}|_{(7)} < 0,\ \varepsilon \in \mathcal{B}_{V_\theta^\alpha,\delta_3}.$$

Since V_θ^β and V_θ^α are homogeneous of degree of r_2 and r_3, respectively, there exists $c_{11} > 0$ such that $V_\theta^\alpha \leqslant c_{11}(V_\theta^\beta)^{\frac{r_3}{r_2}}$. Therefore, $\mathcal{B}_{V_\theta^\beta,\delta_4} \subset \mathcal{B}_{V_\theta^\alpha,\delta_3}$, where $\delta_4 = (\frac{1}{c_{11}})^{\frac{r_2}{r_3}}\delta_3^{r_2/r_3}$. Note that $\underline{\beta}_1 < 0$ and $\underline{\beta}_2 < \beta - 1$. Then, $\lim\limits_{\theta\to\infty}|\delta_2| = \lim\limits_{\theta\to\infty}\delta_3 = \lim\limits_{\theta\to\infty}\delta_4 = 0$. Thus, for any $\epsilon_3 > 0$, there exists θ_4 such that when $\theta > \theta_4$, $|\delta_2 - \delta_4| < \epsilon_3$. $\mathcal{B}_{V_\theta^\beta,\delta_2}$ and $\mathcal{B}_{V_\theta^\beta,\delta_4}$ are close to each other. Note that $\frac{\mathrm{d}V_\theta^\alpha(\varepsilon)}{\mathrm{d}t}|_{(7)}$ is continuous on $\mathbb{R}^n$. Then, we can choose $\theta > \max\{\theta_i\}$ $(i = 1, 2, 3, 4)$, such that $\frac{\mathrm{d}V_\theta^\alpha(\varepsilon)}{\mathrm{d}t}|_{(7)} < 0$, $\varepsilon \in \mathcal{B}_{V_\theta^\beta,\delta_2}\backslash\mathcal{B}_{V_\theta^\beta,\delta_4}$. The proof is completed.

Using the same method as Theorem 1, we can extend the results to the system (1) with the following condition

$$|f_i(y, x_2, \cdots, x_i, u) - f_i(y, \hat{x}_2, \cdots, \hat{x}_i, u)| \leqslant \sum_{j=2}^{i} l_{1,ij}|x_j - \hat{x}_j|^{p_{1,ij}} + \sum_{j=2}^{i} l_{3,ij}|x_j - \hat{x}_j|^{p_{2,ij}} + \varGamma(u, y)(l_{2,i} + \sum_{j=2}^{n}|\hat{x}_j|^{v_j})\sum_{j=2}^{i}|x_j - \hat{x}_j|, \tag{18}$$

where $l_{1,ij}$, $l_{2,i}$, $l_{3,ij}$, $p_{1,ij}$ and $p_{2,ij}$ are given in (2).

Theorem 2 If $\frac{n-i}{n-j+1} < p_{1,ij} < 1$, $1 < p_{2,ij} < \frac{i}{j-1}$ $(2 \leqslant j \leqslant i \leqslant n)$, for any α, $1 - \frac{1}{n} < \alpha < 1$, any σ, $0 < \sigma < \min\{\frac{i - p_{2,ij}(j-1)}{1+p_{2,ij}}, \frac{i - p_{2,ij}(j-1)}{2(p_{2,ij}-1)}, 1\}$, and any β, $1+\dfrac{1}{2\sigma}<\beta<1+\dfrac{1}{\sigma}$, there exist $\varphi_i>0$ $(i=1,2,3)$ such that a global asymptotic observer of the system (1) with the condition (18) can be designed as follows:

$$\begin{cases}\dot{\hat{x}}_1=\hat{x}_2+La_1\lceil e_1\rfloor^{\alpha_1}+La_1\lceil e_1\rfloor^{\beta_1}+f_1(y,u),\\ \dot{\hat{x}}_2=\hat{x}_3+La_2\lceil e_1\rfloor^{\alpha_2}+La_2\lceil e_1\rfloor^{\beta_2}+\\ \qquad f_2(y,\hat{x}_2,u),\\ \quad\vdots\\ \dot{\hat{x}}_n=La_n\lceil e_1\rfloor^{\alpha_n}+La_n\lceil e_1\rfloor^{\beta_n}+\\ \qquad f_n(y,\hat{x}_2,\cdots,\hat{x}_n,u),\\ \dot{L}=-L(\varphi_1(L^{1-\sigma}-\varphi_2)-\varphi_3\Gamma(u,y)(l_{2,i}+\\ \qquad \sum\limits_{j=2}^{n}|\hat{x}_j|^{v_j}),L(0)>\varphi_2,\end{cases} \tag{19}$$

where a_i $(1\leqslant i\leqslant n)$ are given such that

$$A^{\mathrm{T}}P+PA\leqslant -I,\ \underline{\lambda}I\leqslant \Lambda P+P\Lambda\leqslant\bar{\lambda}I, \tag{20}$$

and $A=\begin{bmatrix}-a_1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n-1} & 0 & \cdots & 1\\ -a_n & 0 & \cdots & 0\end{bmatrix}$, $\Lambda=\mathrm{diag}\{\sigma,1+\sigma,\cdots,n-1+\sigma\}$, $\bar{\lambda}$, $\underline{\lambda}$ are two real contants, α_i and β_i are given by (5), $\Gamma(u,y)$ is a continuous function.

Remark 2 Similar system to (1) with the condition (2) has been considered in [14]. In fact, the authors in [14] allowed $f_i(\cdot)$ in (2) to have both low-order and high-order nonlinearities

$$|f_i(x_1,\cdots,x_i)|\leqslant\\ c(|x_1|^{m_{i+1}/m_1}+\cdots+|z_i|^{m_{i+1}/m_i}+\\ |x_1|^{r_{k+1}/r_1}+\cdots+|x_1|^{r_{k+1}/r_k})$$

for a constant $c>0$ with m_i and r_i defined as $m_1=r_1=1$, $m_{i+1}=m_i+\tau_2$, $r_{i+1}=r_i+\tau_1$ for $i=1,\cdots,n$, $\tau_1\geqslant 0$, $-\dfrac{1}{n}<\tau_2\leqslant 0$. Then, a dual-observer design for the global output feedback stabilization of system (1) was developed. The proof of the global asymptotical stability is based on the recursive method.

Remark 3 In [10], the authors presented a class of high gain observers for the system (1) with the following condition

$$|f_i(y,x_2,\cdots,x_i,u)-f_i(y,\hat{x}_2,\cdots,\hat{x}_i,u)|\leqslant\\ \Gamma(u,y)(1+\sum_{j=2}^{n}|\hat{x}_j|^{v_j})\sum_{j=2}^{i}|x_j-\hat{x}_j|+\\ l\sum_{j=2}^{i}|x_j-\hat{x}_j|^{\frac{1-d(n-i-1)}{1-d(n-j)}}, \tag{21}$$

where $\Gamma(\cdot)$ is a continuous function, $v_j\in[0,\dfrac{1}{j-1})$ $(j=2,\cdots,n)$ and $0\leqslant d<\dfrac{1}{n-1}$. The condition (21) results in homogeneity in the bi-limit which is useful for the high gain observer design in a global way and the lower bound of $\dfrac{1-d(n-i-1)}{1-d(n-j)}$ is greater than 1. In fact, (21) is a special case of (18), because if we set that $p_{2,ij}=\dfrac{1-d(n-i-1)}{1-d(n-j)}$, $l_{1,ij}=0$, $l_{2,i}=1$, $l_{3,ij}=l$, then $1<p_{2,ij}<\dfrac{i}{j-1}$.

Remark 4 The observer design were done recursively together with the appropriate error Lyapunov functions which are also constructed by recursive procedures in [10, 14]. Whereas, the observer designed in our paper has an explicit form. The design parameters $[S]^{-1}_{i1}$, a_i can be easily determined by (6) and (20), respectively.

3 Numerical simulation

In this section, we give two examples to demonstrate the effectiveness of our method.

Example 1 We give the following observer design for the system (3)

$$\begin{cases}\dot{\hat{x}}_1=\hat{x}_2+\theta_4\lceil e_1\rfloor^{\alpha_1}+\theta_4\lceil e_1\rfloor^{\beta_1},\\ \dot{\hat{x}}_2=\hat{x}_3-\hat{x}_1-\hat{x}_2+\theta_6\lceil e_1\rfloor^{\alpha_2}+\theta_6\lceil e_1\rfloor^{\beta_2},\\ \dot{\hat{x}}_3=\hat{x}_4-\hat{x}_2+\theta_4\lceil e_1\rfloor^{\alpha_3}+\theta_4\lceil e_1\rfloor^{\beta_3},\\ \dot{\hat{x}}_4=\theta\lceil e_1\rfloor^{\alpha_4}+\theta\lceil e_1\rfloor^{\beta_4}+f_4(x_1,\hat{x}_2,\hat{x}_3,\hat{x}_4),\end{cases}$$

where $\alpha_i=i\alpha-(i-1)$, $\beta_i=i\beta-(i-1)$, $1\leqslant i\leqslant 4$, $\alpha=0.9$, $\varphi_1=0.9$, $\varphi_2=1.1$, $\varphi_3=0.01$, $\theta=5$. Let $\sigma=0.5$, $\beta=2.2$, we have $p_{2,42}=2<\dfrac{4-0.5}{2-1+0.5}$ and $\sigma<\dfrac{4-2(2-1)}{2+1}$, $\sigma<\dfrac{4-2(2-1)}{2(2-1)}$ and $1+\dfrac{1}{2\times 0.5}<\beta<1+\dfrac{1}{0.5}$.

The initial conditions of (3) and its observer are given by $(2,-14,-8,3)^{\mathrm{T}}$ and $(-7,10,11,-4)^{\mathrm{T}}$, respectively. The simulation can be performed by employing integration of ODE45. Fig. 2 and Fig. 3 show the trajectories of the system states and the error states, respectively. The simulation results with noisy measurement is shown in Fig.4.

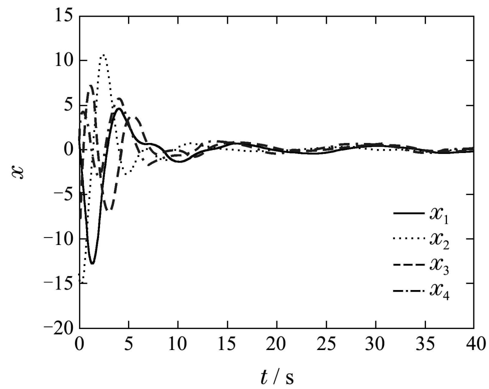

Fig. 2 The trajectories of the system states

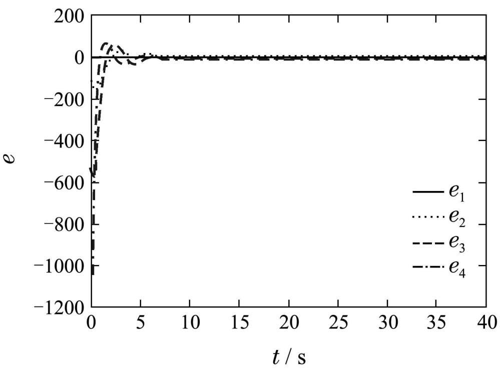

Fig. 3 The trajectories of the error states

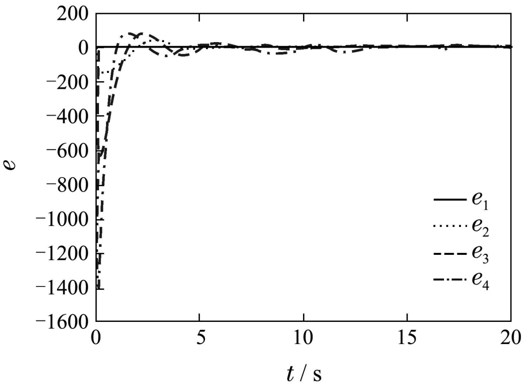

Fig. 4 The simulation results with noisy measurement (a band-limited white noise with noise power 0.1)

Example 2 A bioreactor is a reactor in which microorganism grow by eating a substrate. Let η_1 and η_2 denote the concentrations of microorganisms and substrate, respectively. We obtain the following equations for the bioreactor[18]:

$$\begin{cases} \dot{\eta}_1 = \dfrac{\eta_1\eta_2}{\eta_1+\eta_2} - u\eta_1, \\ \dot{\eta}_2 = -\dfrac{\eta_1\eta_2}{\eta_1+\eta_2} + u(1-\eta_2), \\ y = \eta_1, \end{cases} \tag{22}$$

where u is the control and is in the interval $\mathcal{M}_u = [u_{\min}, u_{\max}] \subset (0,1)$. Note that the following set is forward invariant[7]:

$$\mathcal{M}_\eta = \left\{(\eta_1,\eta_2) \in \mathbb{R}^2 : \eta_1 \geqslant \epsilon_1, \eta_2 \geqslant \epsilon_2, \eta_1+\eta_2 \leqslant 1\right\},$$

where $\epsilon_1 = \dfrac{(1-u_{\max})\epsilon_2}{u_{\max}}$, and $u_{\min} \geqslant \epsilon_2$. This ensures that the bioreactor state remains in a known compact set.

Consider the following coordinate transformation

$$(\eta_1,\eta_2) \to (x_1,x_2) = F(\eta_1,\eta_2) = (\eta_1, \frac{\eta_1\eta_2}{\eta_1+\eta_2}).$$

Then, we obtain

$$\begin{cases} \dot{x}_1 = x_2 - ux_1, \\ \dot{x}_2 = f_2(x_1,x_2,u), \\ y = x_1, \end{cases} \tag{23}$$

where $f_2(x_1,x_2,u) = m_0 + m_1x_2 + m_2x_2^2$, $m_0 = u$, $m_1 = -u-1-\dfrac{2u}{x_1}$, $m_2 = \dfrac{2}{x_1} + \dfrac{u}{x_1^2}$. Let $\mathcal{M}_x$ denote the set in which x evolves. Then, $\mathcal{M}_x = F(\mathcal{M}_\eta)$. We have

$$\underline{x}_2(x_1) = x_1\frac{\epsilon_2}{x_1+\epsilon_2} \leqslant x_2 \leqslant x_1(1-x_1) = \bar{x}_2(x_1).$$

As in [10], let

$$x_{1s} = \max\{\epsilon_1, \min\{1-\epsilon_2, x_1\}\},$$
$$x_{2s} = \max\{\underline{x}_2(x_{1s}), \min\{\bar{x}_2(x_{1s}), x_2\}\},$$
$$\hat{x}_{2s} = \max\{\underline{x}_2(x_{1s}), \min\{\bar{x}_2(x_{1s}), \hat{x}_2\}\},$$

$$\Omega(u,x_1,\hat{x}_2) = \max_{x_2\in[\underline{x}_2(1s),\bar{x}_2(x_{1s})]} |m_1 + m_2\hat{x}_2^p(\hat{x}_2^{1-p} + x_2^{1-p})|,$$

$$l_3 = \max_{(x_1,x_2,\hat{x}_2)\in\mathcal{M}_x\times[\underline{x}_2(\epsilon_1),\bar{x}_2(1-\epsilon_2)]} |m_2x_2^{1-p}|,$$

where $0<p<1$. Then,

$$|f_1(x_1,x_2,u) - f_2(x_1,\hat{x}_2,u)| < \Omega(u,x_1,\hat{x}_2)|x_2-\hat{x}_2| + l_3|x_2-\hat{x}_2|^{1+p}.$$

We construct the following observer for (23)

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 - u\hat{x}_1 + 3L\lceil e_1\rfloor^{\alpha_1} + 3L\lceil e_1\rfloor^{\beta_1}, \\ \dot{\hat{x}}_2 = f_2(x_1,\hat{x}_2,u) + 2L\lceil e_1\rfloor^{\alpha_2} + 2L\lceil e_1\rfloor^{\beta_2}, \\ \dot{L} = -L(\varphi_1(L^{1-\sigma}-\varphi_2) - \varphi_3\Omega(u,x_1,\hat{x}_2)), \end{cases} \tag{24}$$

where $e_1 = x_1 - \hat{x}_1$, $\alpha_i = i\alpha-(i-1)$, $\beta_i = i\beta-(i-1)$, $i = 1,2$, $\alpha = 0.9$, $\varphi_1 = 0.8$, $\varphi_2 = 1.5$, $\varphi_3 = 0.01$, the control input $u = 0.4$. Let $p = 0.6$, $\sigma = 0.1$, $\beta = 7$, then $p_{2,22} = 1.6$, $\sigma < \dfrac{2-1.6(2-1)}{1+1.6}$, $\sigma < \dfrac{2-1.6(2-1)}{2(1.6-1)}$, and $1+\dfrac{1}{0.2} < \beta < 1+\dfrac{1}{0.1}$.

By Theorem 1 in [10], a global asymptotic stable observer is given as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 - u\hat{x}_1 + L^{1.4}q_1(l_1\dfrac{e_1}{L^{0.4}}), \\ \dot{\hat{x}}_2 = f_2(x_1,\hat{x}_2,u) + L^{2.4}q_2(l_2q_1(l_1\dfrac{e_1}{L^{0.4}})), \\ \dot{L} = -L(\varphi_1(L-\varphi_2) - \varphi_3\Omega(u,x_1,\hat{x}_2)) \end{cases} \tag{25}$$

where $q_1(s) = s + s^{1/(1-p)}$, $q_2(s) = s + s^{1+p}$, $\varphi_1 = 0.03$, $\varphi_2 = 1$, $\varphi_3 = 3$, $l_1 = l_2 = 0.01$.

The initial conditions of (23) and (24), (25) are given by $(0.7, 0.3)$ and $(0.3, 0.5)$, respectively. Fig. 5 shows the simulation results. The trajectories of the states obtained by the observer (24) converge faster in transient than the ones obtained by (25).

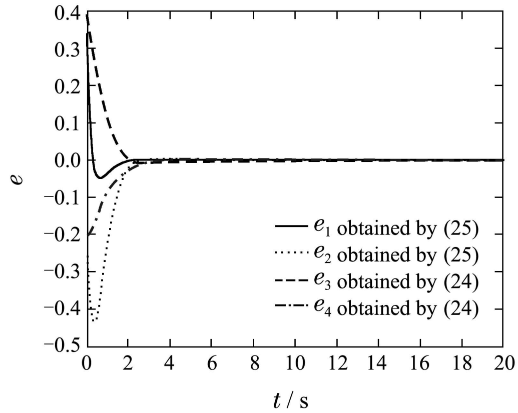

Fig. 5 The trajectories of the error states

4 Conclusions

In this paper, global asymptotic stable observers are proposed for a class of systems with uniform observability. The characteristic of the class of systems is the non-Lipschitz type of conditions with mixed rational powers ($\dfrac{n-i}{n-j-1} < p_{1,ij} < 1$, $1 < p_{2,ij} < \dfrac{i}{j-1}$) of the increments. Two numerical examples were given to show the validity of our method.

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Appendix: Proof of Proposition 1

By an obvious change of integration, we can easily verify i).

Let

$$\mathcal{S}_1 = \{\varepsilon : \varepsilon^{\mathrm{T}}\varepsilon = 1\},$$
$$\bar{\mathcal{B}}_{1,\delta} \triangleq \{\varepsilon : \varepsilon^{\mathrm{T}}\varepsilon \leqslant \delta\},\ \mathcal{B}_{1,\delta} \triangleq \{\varepsilon : \varepsilon^{\mathrm{T}}\varepsilon < \delta\},$$
$$\mathcal{B}_{2,\delta} \triangleq \{\varepsilon : \sum_{i=2}^{n}\varepsilon_i^2 < \delta^2\},\ \bar{\mathcal{B}}_{2,\delta} \triangleq \{\varepsilon : \sum_{i=2}^{n}\varepsilon_i^2 \leqslant \delta^2\},$$
$$\mathcal{B}'_{2,\delta,\sigma} \triangleq \{(\theta^{-\sigma\lambda_1}\varepsilon_2, \cdots, \theta^{-\sigma\lambda_{n-1}}\varepsilon_n)^{\mathrm{T}}, \varepsilon \in \mathcal{B}_{2,\delta}\},$$
$$\bar{\mathcal{B}}'_{2,\delta,\sigma} \triangleq \{(\theta^{-\sigma\lambda_1}\varepsilon_2, \cdots, \theta^{-\sigma\lambda_{n-1}}\varepsilon_n)^{\mathrm{T}}, \varepsilon \in \bar{\mathcal{B}}_{2,\delta}\},$$
$$\bar{\mathcal{P}}_{\delta} \triangleq \{\varepsilon : |\varepsilon_1| \leqslant \delta\},$$

and $\mathcal{P}_\delta \triangleq \{\varepsilon : |\varepsilon_1| < \delta\}$, $\mathcal{F}_\delta \triangleq \{\varepsilon : |\varepsilon_1| = \delta\}$, where $\delta > 0$. The proof is divided into four parts. The first three parts are to construct a compact set containing the origin and the homogeneous Lyapunov function (9) satisfies some inequalities on this set. Then, these inequalities imply (10). The illustrative diagram is given as follows.

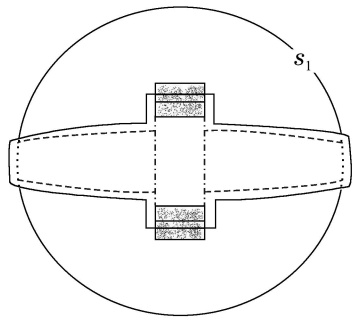

Fig. A The illustrative diagram of the compact set

Firstly, we shall prove

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < -\theta d_1,\ \varepsilon \in \mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-\sigma}}\text{(dotted line)}, \quad \text{(A1)}$$

for a constant $d_1 > 0$. Secondly, we shall prove that there exist $\delta_1, d_2 > 0$, the inequality

$$\begin{cases} \dfrac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < -\theta^{1-r_1\sigma}d_2, \\ \varepsilon \in (\bar{\mathcal{P}}_{(1+\delta_1)\theta^{-\sigma}} \backslash \mathcal{P}_{(1-\delta_1)\theta^{-\sigma}}) \cap \\ \quad \bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \text{ (shaded area)} \end{cases} \quad \text{(A2)}$$

holds, where $\sigma_2 = \sigma + \sigma_1$, $\sigma_1 = \max\{\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_{n-1}}\}(-(\lambda-1)\sigma+2)$. Thirdly, we shall prove that there exist $\tilde{\theta} > 1$, $h > \max\{1, \dfrac{\sigma_2}{\sigma} - 1\}$, $d_3 > 0$ such that

$$\begin{aligned} &\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < -\theta^{1-(r_1+\lambda-1)\sigma_2}\tilde{\theta}^{(r_1+\lambda-1)\sigma}d_3, \\ &\forall\varepsilon \in \mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1} \backslash \mathcal{B}'_{2,\delta_1,\sigma_2})\text{(dashed line).} \end{aligned} \quad \text{(A3)}$$

Note that $\dfrac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}$ is continuous on $\mathbb{R}^n$, we can obtain $\dfrac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < 0$, $\forall\varepsilon \in (\bar{\mathcal{P}}_{\theta^{-\sigma}} \backslash \mathcal{P}_{\theta^{-(1+h)\sigma}}) \cap (\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \cap (\bar{\mathcal{B}}_{1,1} \backslash \mathcal{B}'_{2,\delta_1,\sigma_2}))$. Then, we obtain a compact set $(\mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-(1+h)\sigma}}) \cup (\mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1} \backslash \mathcal{B}'_{2,\delta_1,\sigma_2})) \cup (\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \cap \mathcal{F}_{\theta^{-\sigma}}) \cup ((\bar{\mathcal{P}}_{\theta^{-\sigma}} \backslash \mathcal{P}_{\theta^{-(1+h)\sigma}}) \cap (\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \cap (\bar{\mathcal{B}}_{1,1} \backslash \mathcal{B}'_{2,\delta_1,\sigma_2})))$ containing 0 such that $\dfrac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < 0$.

When the compact set is derived, it is easy to prove the following inequalities

$$V_\theta^\lambda(\varepsilon)^{-\gamma_1} > d_4^{-\gamma_1}, \forall\varepsilon \in \mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-(1+h)\sigma}}, \tag{A4}$$

$$V_\theta^\lambda(\varepsilon)^{-\gamma_1} > d_5^{-\gamma_1}\theta^{(r_1+\lambda-1)\sigma},\ \forall\varepsilon\in\mathcal{F}_{\theta^{-\sigma}}\cap\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2}, \tag{A5}$$

$$\begin{aligned}&V_\theta^\lambda(\varepsilon)^{-\gamma_1} > d_6^{-\gamma_1}\theta^{r_1\sigma_2\gamma_1}\tilde{\theta}^{-r_1\sigma\gamma_1},\\&\forall\varepsilon \in \mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})\end{aligned} \tag{A6}$$

hold for three constants d_4, d_5 and $d_6 > 0$. Then, for any $\varepsilon \in \mathbb{R}^n$, there exist $v_0 > 0$, $\varepsilon_0 \in (\mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-(1+h)\sigma}})\cup(\mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2}))\cup(\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2}\cap\mathcal{F}_{\theta^{-\sigma}})\cup((\bar{\mathcal{P}}_{\theta^{-\sigma}}\backslash\mathcal{P}_{\theta^{-(1+h)\sigma}})\cap(\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2}\cap(\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})))$ such that $\varepsilon = [\varepsilon_1\ \cdots\ \varepsilon_n]^{\mathrm{T}} = [v_0\varepsilon_{01}\ \cdots\ v_0^{\lambda_{n-1}}\varepsilon_{0n}]^{\mathrm{T}}$, and

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} = v_0^{r_1+\lambda-1}\frac{\mathrm{d}V_\theta^\lambda(\varepsilon_0)}{\mathrm{d}t}|_{(8)}.$$

At the same time, note that $V_\theta^\lambda(\varepsilon) = v_0^{r_1}V_\theta^\lambda(\varepsilon_0)$. Thus,

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} = -V_\theta^\lambda(\varepsilon_0)^{-\gamma_1}\frac{\mathrm{d}V_\theta^\lambda(\varepsilon_0)}{\mathrm{d}t}|_{(8)}V_\rho^\lambda(\varepsilon)^{\gamma_1}.$$

The following inqualities also hold

$$1 + (\lambda - 1)\sigma > 1 - \sigma, \text{for } 1 - \frac{1}{n} < \lambda < 1, \tag{A7}$$

$$1 + (\lambda - 1)\sigma > 1, \text{for } 1 \leqslant \lambda < 1 + \frac{1}{\sigma}. \tag{A8}$$

By using the inequalities (A1)–(A8) and the continuity of $\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}$ on $\mathbb{R}^n$, we can obtain (10) and (11).

Now, we prove these inequalities hold. Firstly, let l_1 be the largest such $l > 0$ that $\max\limits_{\{v\leqslant l\}}\max\limits_{\{\varepsilon\in\bar{\mathcal{B}}_{1,2}\backslash\mathcal{B}_{1,1/2}\}}\bar{V}(v\varepsilon_1, \cdots, v^{\lambda_{n-1}}\varepsilon_n) \leqslant 1$. Let l_2 be the smallest such $l > 0$ that $\min\limits_{\{v\geqslant l\}}\min\limits_{\{\varepsilon\in\bar{\mathcal{B}}_{1,2}\backslash\mathcal{B}_{1,1/2}\}}\bar{V}(v\varepsilon_1, \cdots, v^{\lambda_{n-1}}\varepsilon_n) \geqslant 2$. Therefore,

$$\begin{aligned}&\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} =\\&2\theta\int_{l_1}^{l_2}\frac{\kappa'(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\cdot\\&K(v, \varepsilon_1, \cdots, \varepsilon_n)\mathrm{d}v,\ \forall\varepsilon \in \bar{\mathcal{B}}_{1,2}\backslash\mathcal{B}_{1,1/2},\end{aligned} \tag{A9}$$

where

$$K(\cdot) = \begin{bmatrix} v\varepsilon_1 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix}^{\mathrm{T}} S \begin{bmatrix} v^{\lambda_1}\varepsilon_2 - \theta^{(\lambda_1-1)\sigma}[S]_{11}^{-1}\lceil v\varepsilon_1\rfloor^{\lambda_1} \\ \vdots \\ -\theta^{(\lambda_n-1)\sigma-n+1}[S]_{n1}^{-1}\lceil v\varepsilon_1\rfloor^{\lambda_n} \end{bmatrix}.$$

If $\varepsilon \in \mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-\sigma}}$, we have

$$\begin{aligned}&\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} =\\&\theta\int_{l_1}^{l_2}\frac{\kappa'(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\cdot\\&\begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix}^{\mathrm{T}}(SA_0 + A_0^{\mathrm{T}}S)\begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_1}\varepsilon_n \end{bmatrix}\mathrm{d}v +\\&2\theta\int_{l_1}^{l_2}\frac{\kappa'(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\cdot\\&(\begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix}^{\mathrm{T}} S \begin{bmatrix} \theta^{-\sigma}[S]_{11}^{-1}v^{\lambda_1} \\ \vdots \\ \theta^{-\sigma-n+1}[S]_{1n}^{-1}v^{\lambda_n} \end{bmatrix} +\\&\begin{bmatrix} \pm v\theta^{-\sigma} \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{\mathrm{T}} S \begin{bmatrix} v^{\lambda_1}\varepsilon_2 - \theta^{-\sigma}[S]_{11}^{-1}v^{\lambda_1} \\ \vdots \\ -\theta^{-\sigma-n+1}[S]_{n1}^{-1}v^{\lambda_n} \end{bmatrix})\mathrm{d}v.\end{aligned}$$

From (6), we have

$$\begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix}^{\mathrm{T}}(SA_0 + A_0^{\mathrm{T}}S)\begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_1}\varepsilon_n \end{bmatrix} \leqslant -\underline{s}\sum_{i=2}^{n}v^{2\lambda_{i-1}}\varepsilon_i^2. \tag{A10}$$

Then, it is easy to be obtained there exists $\theta_1 > 1$ such that when $\theta > \theta_1$, we have

$$\begin{aligned}&\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} <\\&-\theta\int_{l_1}^{l_2}\frac{\underline{s}\sum\limits_{i=2}^{n}v^{2\lambda_{i-1}}\varepsilon_i^2}{2v^{r_1+\lambda}}\kappa'(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))\mathrm{d}v.\end{aligned}$$

$\mathcal{S}_1 \cap \bar{\mathcal{P}}_0 \subset \mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-\sigma}} \subset \mathcal{S}_1 \cap \bar{\mathcal{P}}_{2^{-\sigma}}$. Let l_3 be the largest such $l > 0$ that $\max_{\{v\leqslant l\}}\max_{\{\varepsilon\in\mathcal{S}_1\cap\bar{\mathcal{P}}_0\}}\bar{V}(v\varepsilon_1, \cdots, v^{\lambda_{n-1}}\varepsilon_n) \leqslant 1$. Let l_4 be the smallest such $l > 0$ that $\min_{\{v\geqslant l\}}\min_{\{\varepsilon\in\mathcal{S}_1\cap\bar{\mathcal{P}}_0\}}\bar{V}(v\varepsilon_1, \cdots, v^{\lambda_{n-1}}\varepsilon_n) \geqslant 2$. Then, $l_3 > l_1$ and $l_4 < l_2$. Therefore,

$$\begin{aligned}&\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} <\\&-\theta\int_{l_3}^{l_4}\frac{\underline{s}\sum\limits_{i=2}^{n}v^{2\lambda_{i-1}}\varepsilon_i^2}{2v^{r_1+\lambda}}\kappa'(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))\mathrm{d}v <\\&-\theta d_1, \varepsilon \in \mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-\sigma}},\end{aligned} \tag{A11}$$

where

$$\begin{aligned}d_1 = \min_{\{\varepsilon\in\mathcal{S}_1\cap\bar{\mathcal{P}}_{2^{-\sigma}}\}}\int_{l_3}^{l_4}\frac{\underline{s}\sum\limits_{i=2}^{n}v^{2\lambda_{i-1}}\varepsilon_i^2}{2v^{r_1+\lambda}}\kappa'\cdot\\(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))\mathrm{d}v.\end{aligned}$$

Secondly, it follows from (A9) and $(\lambda_1 - 1)\sigma > \max\{(\lambda_j - 1)\sigma - j + 1\}$ $(2 \leqslant j \leqslant n)$ that there exist $\theta_2 > 1$, $\delta_1 > 0$ such that when $\theta > \theta_2$, we have

$$\begin{aligned}&\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} <\\&-2\theta^{(\lambda-1)\sigma+1}\int_{l_1}^{l_2}\frac{\kappa'(\bar{V}(v\varepsilon_1, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\cdot\\&\underline{s}[S]_{11}^{-1}|v\varepsilon_1|^{1+\lambda_1})\mathrm{d}v -\\&2\theta\int_{l_1}^{l_2}\frac{\kappa'(\bar{V}(v\varepsilon_1, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix} S\cdot\end{aligned}$$

$$\begin{bmatrix} \theta^{(\lambda_1-1)\sigma}[S]^{-1}_{11}\lceil v\varepsilon_1\rfloor^{\lambda_1} \\ \vdots \\ \theta^{(\lambda_n-1)\sigma-n+1}[S]^{-1}_{n1}\lceil v\varepsilon_1\rfloor^{\lambda_n} \end{bmatrix} \mathrm{d}v +$$

$$2\theta\int_{l_1}^{l_2} \frac{\kappa'(\bar V(v\varepsilon_1,\cdots,v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}} \begin{bmatrix} v\varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} S\cdot$$

$$\begin{bmatrix} v^{\lambda_1}\varepsilon_2 \\ v^{\lambda_2}\varepsilon_3-\theta^{(\lambda_2-1)\sigma}[S]^{-1}_{21}\lceil v\varepsilon_1\rfloor^{\lambda_2} \\ \vdots \\ \theta^{(\lambda_n-1)\sigma-n+1}[S]^{-1}_{n1}\lceil v\varepsilon_1\rfloor^{\lambda_n} \end{bmatrix} \mathrm{d}v <$$

$$-\theta^{(\lambda-1)\sigma+1}\int_{l_1}^{l_2}\frac{\kappa'(\bar V(\pm v,0,\cdots,0))\underline{s}[S]^{-1}_{11}}{v^{r_1+\lambda}}\mathrm{d}v,$$

$$\varepsilon\in(\bar{\mathcal P}_{1+\delta_1}\backslash\mathcal P_{1-\delta_1})\cap\bar{\mathcal B}'_{2,\delta_1,\sigma_1},$$

where $\sigma_1=\max\{\frac{1}{\lambda_1},\frac{1}{\lambda_{n-1}}\}((-(\lambda-1)\sigma)+2)$.

Note that $\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}$ is homogenous of degree $r_1+\lambda-1$. Then, we have

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}<-d_1\theta^{1-r_1\sigma},$$
$$\varepsilon\in(\bar{\mathcal P}_{(1+\delta_1)\theta^{-\sigma}}\backslash\mathcal P_{(1-\delta_1)\theta^{-\sigma}})\cap\bar{\mathcal B}'_{2,\delta_1,\sigma_2}, \tag{A12}$$

where $d_2=\int_{l_1}^{l_2}\frac{\kappa'(\bar V(\pm v,0,\cdots,0))}{v^{r_1+\lambda}}\underline{s}[S]^{-1}_{11}\mathrm{d}v$, $\sigma_2=\sigma+\sigma_1$.

Thirdly, let l_5 be the largest such $l>0$ that $\max\limits_{\{v\leqslant l\}}\max\limits_{\{\varepsilon\in(\bar{\mathcal P}_{(1+\delta_1)\theta^{-\sigma}}\cap(\bar{\mathcal B}_{1,1}\backslash\mathcal B'_{2,\delta_1,\sigma_2}))\}}\bar V(v\varepsilon_1,\cdots,v^{\lambda_{n-1}}\varepsilon_n)\leqslant 1$. Similarly, let l_6 be the smallest such $l>0$ that

$$\min_{\{v\geqslant l\}}\min_{\{\varepsilon\in\bar{\mathcal P}_{(1+\delta_1)\theta^{-\sigma}}\cap(\bar{\mathcal B}_{1,1}\backslash\mathcal B'_{2,\delta_1,\sigma_2})\}}\bar V(v\varepsilon_1,\cdots,v^{\lambda_{n-1}}\varepsilon_n)\geqslant 2.$$

Thus, for any $\varepsilon\in\bar{\mathcal P}_{(1+\delta_1)\theta^{-\sigma}}\cap(\bar{\mathcal B}_{1,1}\backslash\mathcal B'_{2,\delta_1,\sigma_2})$, we have

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}=$$
$$2\int_{l_5}^{l_6}\frac{\kappa'(\bar V(v\varepsilon_1,v^{\lambda_1}\varepsilon_2,\cdots,v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}K(v,\varepsilon_1,\cdots,\varepsilon_n)\mathrm{d}v.$$

For any $\varepsilon\in\bar{\mathcal P}_{(1+\delta_1)\theta^{-\sigma}}\cap(\bar{\mathcal B}_{1,1}\backslash\mathcal B'_{2,\delta_1,\sigma_2})$, there exists $\tilde\theta(\geqslant 1)$ such that

$$\varepsilon=(\tilde\theta^\sigma\tilde\theta^{-\sigma}\varepsilon_1,\cdots,\tilde\theta^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n)^{\mathrm T},$$
$$|\varepsilon_1|\leqslant(1+\delta_1)\theta^{-\sigma},\ \ \sum_{i=2}^n\varepsilon_i^2=\delta_1^2. \tag{A13}$$

Then, for any

$$(\pm\tilde\theta^\sigma\tilde\theta^{-\sigma}\theta^{-(1+h)\sigma},\tilde\theta^{\lambda_1\sigma}\theta^{-\lambda_1\sigma_2}\varepsilon_2,\cdots,$$
$$\tilde\theta^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n)^{\mathrm T}\in\mathcal F_{\theta^{-(1+h)\sigma}}\cap$$
$$(\bar{\mathcal B}_{1,1}\backslash\mathcal B'_{2,\delta_1,\sigma_2}),\ \ \sum_{i=2}^n\varepsilon_i^2=\delta_1^2,$$

we have

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}<$$
$$-\theta\int_{l_5}^{l_6}(\underline{s}\sum_{i=2}^n v^{2\lambda_{i-1}}\tilde\theta^{2\lambda_{i-1}\sigma}\theta^{-2\lambda_{i-1}\sigma_2}\varepsilon_i^2-$$
$$2\sum_{i=2}^n\sum_{j=1}^n\theta^{1-\sigma-j-h\lambda_j\sigma-\lambda_{i-1}\sigma}\tilde\theta^{\lambda_{i-1}\sigma}v^{\lambda_{i-1}+\lambda_j}\varepsilon_i[S]_{ij}[S]^{-1}_{j1}-$$
$$2\theta^{-(1+h)\sigma_2}\sum_{j=2}^n v^{1+\lambda_{j-1}}[S]_{1,j-1}\tilde\theta^{\lambda_{j-1}\sigma}\theta^{1-\lambda_{j-1}\sigma_2}\varepsilon_j-$$
$$2\sum_{j=2}^n\theta^{1-(h+2)\sigma-h\lambda_j\sigma-j}v^{1+\lambda_j}[S]_{1j}[S]^{-1}_{j1})\cdot$$
$$\frac{\kappa'(\bar V(\pm v\theta^{-(1+h)\sigma},\cdots,v^{\lambda_{n-1}}\tilde\theta^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n))}{v^{r_1+\lambda}}\mathrm{d}v.$$

Note that $\{z:1\leqslant z^{\mathrm T}Sz\leqslant 2\}$ is a bounded set, then, there exist $M_1,M_2>0$ such that

$$M_1<\sum_{j=2}^n z_j^{2/\lambda_{j-1}}<M_2.$$

Then, for $l_5\leqslant v\leqslant l_6$, $\sum\limits_{j=2}^n\varepsilon_j^2=\delta_1$, we have

$$M_1<v^2\tilde\theta^{2\sigma}\theta^{-2\sigma_2}(\tilde\theta^{-2\sigma}\theta^{-2(1+h)\sigma-2\sigma_2}+$$
$$\sum_{j=1}^n\varepsilon^{2/\lambda_{j-1}})<M_2.$$

Note that $\tilde\theta>1$ and $(1+h)\sigma>\sigma_2$, then there exists $\theta_3>1$ such that

$$M_1/(2\sum_{j=1}^n\varepsilon^{2/\lambda_{j-1}})<v^2\tilde\theta^{2\sigma}\theta^{-2\sigma_2}<2M_2/(\sum_{j=1}^n\varepsilon^{2/\lambda_{j-1}}),$$

when $\theta>\theta_3$. Therefore, there exists $\theta_4>1$ such that when $\theta>\theta_4$, we have

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}<$$
$$-\theta\int_{l_5}^{l_6}\frac{\underline{s}\sum\limits_{i=2}^n v^{2\lambda_{i-1}}\varepsilon_i^2\kappa'(\bar V(v\varepsilon_1,\cdots,v^{\lambda_{n-1}}\varepsilon_n))}{2v^{r_1+\lambda}}\cdot$$
$$K(v,\cdots,\varepsilon_n)\mathrm{d}v,\ \varepsilon\in\mathcal F_{\theta^{-(1+h)\sigma}}\cap(\bar{\mathcal B}_{1,1}\backslash\mathcal B'_{2,\delta_1,\sigma_2}).$$

Moreover, for any $\varepsilon\in\mathcal F_{\theta^{-(1+h)\sigma}}\cap(\bar{\mathcal B}_{1,1}/\mathcal B'_{2,\delta_1,\sigma_2})$, let $l_5(\varepsilon)$ and $l_6(\varepsilon)$ be such that $5/4\leqslant\bar V(v\varepsilon_1,\cdots,v^{\lambda_{n-1}}\varepsilon_n)\leqslant 7/4$, $l_5(\varepsilon)\leqslant l\leqslant l_6(\varepsilon)$ (without loss of generality, we assume $l_5(\varepsilon)<l_6(\varepsilon)$). Then

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}<-\frac{5\theta\underline{s}\kappa}{16\bar s}\int_{l_5(\varepsilon)}^{l_6(\varepsilon)}\frac{1}{v^{r_1+\lambda}}\mathrm{d}v=$$
$$-\frac{5\theta\underline{s\kappa}}{16\bar s(r_1+\lambda-1)}\frac{l_6(\varepsilon)^{r_1+\lambda-1}-l_5(\varepsilon)^{r_1+\lambda-1}}{l_6(\varepsilon)^{r_1+\lambda-1}l_5(\varepsilon)^{r_1+\lambda-1}}, \tag{A14}$$

where $\bar s=\lambda_{\max}(S)$.

Note that $\{z:z^{\mathrm T}Sz=\frac{5}{4}\}\cap\{z:z^{\mathrm T}Sz=\frac{7}{4}\}=\varnothing$, there exists $M_3>0$ such that

$$M_3<\sum_{i=1}^n(z_i^{1\,(r_1+\lambda-1)/\lambda_{i-1}}-z_i^{2\,(r_1+\lambda-1)/\lambda_{i-1}})^2,$$

where $z^1=(z_1^1,\cdots,z_n^1)\in\{z:z^{\mathrm T}Sz=\frac{7}{4}\}$ and $z^2=(z_1^2,\cdots,z_n^2)\in\{z:z^{\mathrm T}Sz=\frac{5}{4}\}$.

$$(l_6(\varepsilon)\tilde\theta^\sigma(\tilde\theta^{-\sigma}\theta^{-(1+h)\sigma}),l_6(\varepsilon)^{\lambda_1}\tilde\theta^{\lambda_1\sigma}\theta^{-\lambda_1\sigma_2}\varepsilon_2,\cdots,$$
$$l_6(\varepsilon)^{\lambda_{n-1}}\tilde\theta^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n)^{\mathrm T}\in\{z:z^{\mathrm T}Sz=\frac{7}{4}\}$$

and

$$(l_5(\varepsilon)\tilde{\theta}^{\sigma}(\tilde{\theta}^{-\sigma}\theta^{-(1+h)\sigma}), l_5(\varepsilon)^{\lambda_1}\tilde{\theta}^{\lambda_1\sigma}\theta^{-\lambda_1\sigma_2}\varepsilon_2, \cdots, l_5(\varepsilon)^{\lambda_{n-1}}\tilde{\theta}^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n)^{\mathrm{T}} \in \{z : z^{\mathrm{T}}Sz = \frac{5}{4}\}.$$

Therefore,

$$M_3 < \sum_{i=2}^{n}((l_6(\varepsilon)^{\lambda_{i-1}}\tilde{\theta}^{\lambda_{i-1}\sigma}\theta^{-\lambda_{i-1}\sigma_2}\varepsilon_i)^{(r_1+\lambda-1)/\lambda_{i-1}} - (l_5(\varepsilon)^{\lambda_{i-1}}\tilde{\theta}^{\lambda_{i-1}\sigma}\theta^{-\lambda_{i-1}\sigma_2}\varepsilon_i)^{(r_1+\lambda-1)/\lambda_{i-1}})^2 + (l_6(\varepsilon)^{r_1+\lambda-1} - l_5(\varepsilon)^{r_1+\lambda-1})^2\theta^{-2(1+h)(r_1+\lambda-1)\sigma} \leqslant \tilde{\theta}^{2(r_1+\lambda-1)\sigma}\theta^{-2(r_1+\lambda-1)\sigma_2}(l_6(\varepsilon)^{r_1+\lambda-1} - l_5(\varepsilon)^{r_1+\lambda-1})^2(\sum_{i=2}^{n}\varepsilon_i^{2(r_1+\lambda-1)/\lambda_{i-1}} + 1),\ \sum_{i=2}^{n}\varepsilon_i^2 = \delta_1^2.$$

On the other hand, $\{z : 1 \leqslant z^{\mathrm{T}}Sz \leqslant 2\}$ is a bounded compact set, then, there exist $M_4, M_5 > 0$ such that

$$M_4 < \sum_{i=2}^{n} z_i^{2(r_1+\lambda_1-1)/\lambda_{i-1}} < M_5,\ z \in \{z : 1 \leqslant z^{\mathrm{T}}Sz \leqslant 2\}.$$

Clearly, there exist $\varepsilon^j \in \mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})$ such that

$$(l_j(\varepsilon)\tilde{\theta}^{\sigma}(\tilde{\theta}^{-\sigma}\theta^{-(1+h)\sigma}),\ l_j(\varepsilon)^{\lambda_1}\tilde{\theta}^{\lambda_1\sigma}\theta^{-\lambda_1\sigma_2}\varepsilon_2^j, \cdots, l_j(\varepsilon)^{\lambda_{n-1}}\tilde{\theta}^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n^j)^{\mathrm{T}} \in \{z : 1 \leqslant z^{\mathrm{T}}Sz \leqslant 2\},\ j = 5, 6.$$

Then

$$M_5 > \sum_{i=2}^{n}(l_j(\varepsilon)^{\lambda_{i-1}}\tilde{\theta}^{\lambda_{i-1}\sigma}\theta^{-\lambda_{i-1}\sigma_2}\varepsilon_i^j)^{2(r_1+\lambda-1)/\lambda_{i-1}} = \tilde{\theta}^{2(r_1+\lambda-1)\sigma}\theta^{-2(r_1+\lambda-1)\sigma_2}l_j(\varepsilon)^{2(r_1+\lambda-1)}\cdot \sum_{i=2}^{n}{\varepsilon_i^j}^{2(r_1+\lambda-1)/\lambda_{i-1}},\ \sum_{i=2}^{n}{\varepsilon_i^j}^2 = \delta_1^2, j = 5, 6.$$

Then

$$l_6(\varepsilon)^{r_1+\lambda-1} - l_5(\varepsilon)^{r_1+\lambda-1} > \min_{\{\varepsilon:\ \sum_{i=2}^{n}\varepsilon_i^2 = \delta_1^2\}}\sqrt{\frac{\theta^{2(r_1+\lambda-1)\sigma_2}M_3}{\tilde{\theta}^{2(r_1+\lambda-1)\sigma}(\sum_{i=2}^{n}\varepsilon_i^{2(r_1+\lambda-1)/\lambda_{i-1}}+1)}}, \tag{A15}$$

$$\begin{cases} \dfrac{1}{l_j(\varepsilon)^{r_1+\lambda-1}} > \min\limits_{\{\varepsilon:\ \sum_{i=2}^{n}\varepsilon_i^2 = \delta_1^2\}}\sqrt{\dfrac{\tilde{\theta}^{2(r_1+\lambda-1)\sigma}\sum_{i=2}^{n}\varepsilon_i^{2(r_1+\lambda-1)/\lambda_{i-1}}}{\theta^{2(r_1+\lambda-1)\sigma_2}M_5}}, \\ j = 5, 6. \end{cases} \tag{A16}$$

Thus, for $\forall\varepsilon \in \mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})$, from (A14)–(A15) and (A16), we have

$$\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < -\theta^{1-(r_1+\lambda-1)\sigma_2}\tilde{\theta}^{(r_1+\lambda-1)\sigma}d_3, \tag{A17}$$

where

$$d_3 = \min_{\{\varepsilon:\ \sum_{i=2}^{n}\varepsilon_i^2 = \delta_1^2\}}\frac{5\underline{s}\kappa\sqrt{M_3}(\sum_{i=2}^{n}\varepsilon_i^{2(r_1+\lambda-1)/\lambda_{i-1}})}{8\bar{s}(r_1+\lambda-1)M_5\sqrt{\sum_{i=2}^{n}\varepsilon_i^{2(r_1+\lambda-1)/\lambda_{i-1}}+1}}.$$

Note that $\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)}$ is continuous on $\mathbb{R}^n$. Then, there exists $\theta_5 > 1$ such that $\theta > \theta_5$ and $\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < 0, \forall\varepsilon \in (\bar{\mathcal{P}}_{\theta^{-\sigma}}\backslash\mathcal{P}_{\theta^{-(1+h)\sigma}}) \cap (\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2}))$. Thus, we obtain a compact set $(\mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-(1+h)\sigma}}) \cup (\mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})) \cup (\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \cap \mathcal{F}_{\theta^{-\sigma}}) \cup ((\bar{\mathcal{P}}_{\theta^{-\sigma}}\backslash\mathcal{P}_{\theta^{-(1+h)\sigma}}) \cap (\bar{\mathcal{B}}'_{2,\delta_1,\sigma_2} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})))$ containing 0 such that $\frac{\mathrm{d}V_\theta^\lambda(\varepsilon)}{\mathrm{d}t}|_{(8)} < 0$.

The discussion can continue for three cases: i) $\varepsilon \in \mathcal{S}_1 \cap \bar{\mathcal{P}}_{\theta^{-(1+h)\sigma}}$; ii) $\varepsilon \in \mathcal{F}_{\theta^{-\sigma}} \cap \bar{\mathcal{B}}'_{2,\delta_1,\sigma_2}$; iii) $\varepsilon \in \mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})$.

i) If $\varepsilon \in \mathcal{S}_1 \cap \mathcal{P}_{\theta^{-(1+h)\sigma}}$, we can select $\theta > \{1, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ such that $V(\varepsilon)^{-\gamma_1} > d_4^{-\gamma_1}, \varepsilon \in \mathcal{S}_1 \cap \mathcal{P}_{\theta^{-(1+h)\sigma}}$, where

$$d_4 = \min_{\sum_{i=2}^{n}\varepsilon_i^2 = 1}\int_{-\infty}^{+\infty}\frac{\kappa(\bar{V}(0, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\mathrm{d}v.$$

ii) If $\varepsilon \in \mathcal{F}_{\theta^{-\sigma}} \cap \bar{\mathcal{B}}'_{2,\delta_1,\sigma_2}$, from (A13), we can obtain

$$V(\pm\theta^{-\sigma}, \theta^{-\lambda_1\sigma_2}\varepsilon_2, \cdots, \theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n) = \theta^{-r_1\sigma}V(\pm 1, \theta^{-\lambda_1\sigma_1}\varepsilon_2, \cdots, \theta^{-\lambda_{n-1}\sigma_1}\varepsilon_n) \leqslant d_5\theta^{-r_1\sigma},$$

where

$$d_5 = \max_{\sum_{i=2}^{n}\varepsilon_i^2 \leqslant \delta_1^2}\int_{-\infty}^{+\infty}\frac{\kappa(\bar{V}(\pm 1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\mathrm{d}v.$$

Therefore, $V(\varepsilon)^{-\gamma_1} > d_5^{-\gamma_1}\theta^{\sigma(r_1+\lambda-1)}$.

iii) If $\varepsilon \in \mathcal{F}_{\theta^{-(1+h)\sigma}} \cap (\bar{\mathcal{B}}_{1,1}\backslash\mathcal{B}'_{2,\delta_1,\sigma_2})$, from $(1+h)\sigma - \sigma_2 > 0$ and (A13), we have

$$V(\pm\tilde{\theta}^{\sigma}\tilde{\theta}^{-\sigma}\theta^{-(1+h)\sigma}, \tilde{\theta}^{\lambda_1\sigma}\theta^{-\lambda_1\sigma_2}\varepsilon_2, \cdots, \tilde{\theta}^{\lambda_{n-1}\sigma}\theta^{-\lambda_{n-1}\sigma_2}\varepsilon_n) \leqslant d_6\tilde{\theta}^{r_1\sigma}\theta^{-r_1\sigma_2},$$

where

$$d_6 = \max_{|\varepsilon_1| \leqslant 1,\ \sum_{i=2}^{n}\varepsilon_i^2 = \delta_1^2}\int_{-\infty}^{+\infty}\frac{\kappa(\bar{V}(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \cdots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{r_1+\lambda}}\mathrm{d}v.$$

Therefore, $V(\varepsilon)^{-\gamma_1} > d_6^{-\gamma_1}\theta^{r_1\sigma_2\gamma_1}\tilde{\theta}^{-r_1\sigma\gamma_1}$. The proof is completed.

作者简介:

沈艳军 (1970–), 男, 教授, 硕士生导师, 主要研究方向为非线性系统、鲁邦控制等, E-mail: shenyj@ctgu.edu.cn;

夏小华 (1964–), 男, 教授, 博士生导师, 主要研究方向非线性系统、能源优化、复杂系统等, E-mail: xxia@postino.up.ac.za.