Article ID: 1000−8152(2010)12−1777−06

A fuzzy Lyapunov approach for constrained T-S fuzzy systems design

LUO Yan-bin[1], JIANG Xie-fu[1], ZHONG Wei-min[2]

(1. Institution of Automation, Hangzhou Dianzi University, Hangzhou Zhejiang 310018, China;
2. Key Laboratory of Advanced Control and Optimization for Chemical Processes,
East China University of Science and Technology, Shanghai 200237, China)

Abstract: A class of discrete-time T-S fuzzy systems subject to input saturation is studied by introducing a fuzzy Lyapunov function. A sufficient condition that ensures the stability at the system origin is derived; and the domain of attraction may be enlarged by applying a fuzzy anti-windup compensator to the considered system. This method avoids the difficulty in seeking a common positive-definite matrix P satisfying all fuzzy rules of this system. Moreover, an iterative optimization algorithm for obtaining the anti-windup compensator gain is given.

Key words: discrete-time Takagi-Sugeno fuzzy systems; fuzzy Lyapunov function; fuzzy anti-windup compensator; domain of attraction

CLC number: TP273　　**Document code:** A

约束T-S模糊系统设计的模糊Lyapunov方法

罗艳斌[1], 姜偕富[1], 钟伟民[2]

(1. 杭州电子科技大学 自动化学院, 浙江 杭州 310018; 2. 华东理工大学 化工过程先进控制与优化重点实验室, 上海 200237)

摘要: 通过引入模糊Lyapunov函数, 研究一类执行器饱和的离散T-S模糊系统. 对系统设计模糊抗积分饱和补偿器, 得到系统稳定的充分条件, 并扩大了系统的吸引域. 这种方法避免了寻求一个满足系统所有模糊规则的公共正定矩阵P. 最后, 抗积分饱和补偿器增益通过迭代优化算法得到.

关键词: 离散Takagi-Sugeno模糊系统; 模糊Lyapunov函数; 模糊抗积分饱和补偿器; 吸引域

1 Introduction

Lyapunov function approach is one of the most popular approaches in stability analysis and synthesis. Quadratic functions are often used to be Lyapunov candidate functions. In [1] piecewise quadratic functions were constructed for Lyapunov functions, however, this type of Lyapunov function may not be continuously differentiable and its level sets may not be convex. Ref. [2, 3] introduced a composite quadratic Lyapunov function, which is continuously differentiable and whose level set is the convex hull of a set of ellipsoids. Recently, based on a duality theory for linear differential inclusions, a particular pair of conjugate Lyapunov functions was presented to study the stability of a type of systems with saturation nonlinearities[4], and a much weaker condition than that obtained in [5] was derived. Ref. [6] presented a systematic and comprehensive analysis of a general system with saturation or deadzone components by using quadratic and nonquadratic Lyapunov functions, and showed that these conditions in view of the nonquadratic Lyapunov functions applied were less conservative than the conditions in view of quadratic Lyapunov functions applied. In [7], a new parallel distributed compensation(PDC) was proposed to fully take the advantage of fuzzy Lyapunov function. The new PDC provides a less conservative results than the ordinary PDC.

In this paper, the anti-windup design for the discrete-time T-S fuzzy systems by the fuzzy Lyapunov function is considered. New sufficient conditions that ensures the stability of the origin of the system is derived using a fuzzy Lyapunov function. The proposed

Received date: 2010−05−06; Revised date: 2010−07−24.
Foundation item: Supported by the National Science Foundation of China(60774058, 60904058, 60804029).

method is proved to be less conservative compared to those derived by using quadratic Lyapunov functions.

Notation: The following notation will be used throughout the paper. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^m$ the m dimensional Euclidean space, $\mathbb{R}^{n\times m}$ the set of all $n\times m$ real matrices. To reduce clutter, $*$ denotes the off-diagonal entries in symmetric matrices.

2 Problem statement

Consider the following discrete-time T-S fuzzy system, the i-th rule is described as:

Rule i: IF $z_1(k)$ is M_1^i and $\cdots$ and $z_p(k)$ is M_p^i, THEN

$$\dot{x}(k+1)=A_ix(k)+B_iu(k), \tag{1}$$

$$y(k)=C_ix(k), \tag{2}$$

where M_j^i $(i=1,2,\cdots,r, j=1,2,\cdots,p)$ is fuzzy set, r is the number of IF-THEN rules, $z_1(t),\cdots,z_n(t)$ are the premise variables, $x(k)\in\mathbb{R}^n$ is the state vector, $u(k)\in\mathbb{R}^m$ is the input vector, $y(k)\in\mathbb{R}^m$ is the output vector. It is assumed in this paper that the premise variables do not depend on the input variables $u(t)$.

By using the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifier, the fuzzy model (1) can be expressed as the following:

$$\begin{cases} x(k+1)=\sum\limits_{i=1}^{r}\alpha_i(z(k))\big(A_ix(k)+B_iu(k)\big),\\ y(k)=\sum\limits_{i=1}^{r}\alpha_i(z(k))C_ix(k), \end{cases} \tag{3}$$

where $\alpha_i(z(k))$ is the membership function of the i-th rule.

To guarantee the stability of the system (3) in the absence of input saturation, we assume that the following fuzzy controller has been designed:

Rule i: IF $z_1(k)$ is M_1^i and $\cdots$ and $z_p(k)$ is M_p^i, THEN

$$\begin{cases} \eta(k+1)=A_{c(i)}\eta(k)+B_{c(i)}y(k),\\ v(k)=C_{c(i)}\eta(k)+D_{c(i)}y(k),\\ \qquad i=1,2,\cdots,r_c. \end{cases} \tag{4}$$

Then, the fuzzy controller is

$$\begin{cases} \eta(k)=\sum\limits_{i=1}^{r_c}\alpha_i(z(k))\big(A_{c(i)}\eta(k)+B_{c(i)}y(k)\big),\\ v(k)=\sum\limits_{i=1}^{r_c}\alpha_i(z(k))\big(C_{c(i)}\eta(k)+D_{c(i)}y(k)\big), \end{cases} \tag{5}$$

where $\eta(k)\in\mathbb{R}^{n_c}$ is the controller state, $y(k)$ is the controller input and $v(k)$ is the controller output.

Assume that the actual control input u is subject to actuator saturation, i.e. $u=\sigma(v)$. The function $\sigma:\mathbb{R}^m\to\mathbb{R}^m$ is the standard saturation function defined as

$$\sigma(v)=[\sigma(v_1)\ \sigma(v_2)\ \cdots\ \sigma(v_m)]^{\mathrm{T}}, \tag{6}$$

where $\sigma(v_i)=\operatorname{sgn} v_i\min\{|v_i|,u_{0(i)}\}$, $u_{0(i)}>0$, $i=1,2,\cdots,m$, are the control bounds. In order to weaken the influence of input saturation, an anti-windup term is added to the controller (5). Then the modified fuzzy compensator has the form

$$\begin{cases} \eta(k+1)=\sum\limits_{i=1}^{r_c}\alpha_i(z(k))\big(A_{c(i)}\eta(k)+B_{c(i)}y(k)\big)+\\ \qquad E_c(\sigma(v(k))-v(k)),\\ v(k)=\sum\limits_{i=1}^{r_c}\alpha_i(z(k))\big(C_{c(i)}\eta(k)+D_{c(i)}y(k)\big). \end{cases} \tag{7}$$

Under the above fuzzy modified compensator, the closed-loop system is

$$\xi(k+1)=\sum_{i=1}^{r}\sum_{j=1}^{r_c}\alpha_i(z(k))\alpha_j(z(k))(\bar{A}_{ij}\xi(k)+\bar{B}_i\sigma(F_{ij}\xi(k))), \tag{8}$$

where

$$\xi(k)=\begin{bmatrix}x(k)\\ \eta(k)\end{bmatrix},$$

$$\bar{A}_{ij}=\begin{bmatrix}A_i & 0\\ B_{c(j)}C_i-E_cD_{c(j)}C_i & A_{c(j)}-E_cC_{c(j)}\end{bmatrix},$$

$$\bar{B}_i=\begin{bmatrix}B_i\\ E_c\end{bmatrix}, F_{ij}=[D_{c(j)}C_i\ \ C_{c(j)}].$$

Without loss of generality, we assume that $r=r_c$. Denote $\xi^+=\xi(k+1),\xi=\xi(k),\alpha_i^+=\alpha_i(z(k+1)),\alpha_i=\alpha_i(z(k))$, the system (8) can be rewritten as

$$\xi^+=\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j(\bar{A}_{ij}\xi+\bar{B}_i\sigma(F_{ij}\xi)). \tag{9}$$

It is easy to obtain the following lemma.

Lemma 1 Given $F_{ij},H_{ij}\in\mathbb{R}^{m\times(n+n_c)}, i,j=1,2,\cdots,r$, and $F=\sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r}\alpha_i\alpha_jF_{ij}$, $0\leqslant\alpha_i,\alpha_j\leqslant1$, $\sum\limits_{i=1}^{r}\alpha_i=1$. For $\xi\in\mathbb{R}^{n+n_c}$, if $x\in\bigcap\limits_{i,j=1}^{r}\mathcal{L}(H_{ij})$, then

$$\sigma(F_{ij}\xi)\in\mathrm{co}\{(E_kF_{ij}+E_k^-H_{ij})\xi:k\in[1,2^m]\},\quad i,j\in[1,r].$$

Correspondingly, $\sigma(F_{ij}\xi)$ can be expressed as

$$\sigma(F_{ij}\xi)=\sum_{k=1}^{2^m}\eta_k(E_kF_{ij}+E_k^-H_{ij})\xi,$$

$$\forall i,j=1,2,\cdots,r, k\in[1,2^m], \tag{10}$$

where $\mathrm{co}\{\cdot\}$ denotes a convex hull of a set and $0 \leqslant \eta_k \leqslant 1, \sum\limits_{k=1}^{2^m}\eta_k = 1$.

Combine (8) and (10), one obtains

$$\xi^+=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{2^m}\alpha_i\alpha_j\eta_k(\bar{A}_{ij}+\bar{B}_i(E_kF_{ij}+E_k^-H_{ij}))\xi= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{2^m}\alpha_i\alpha_j\eta_k\hat{A}_{ijk}\xi = \sum_{k=1}^{2^m}\eta_k\hat{A}_k(\alpha)\xi, \tag{11}$$

where

$$\hat{A}_{ijk} = \bar{A}_{ij} + \bar{B}_i(E_kF_{ij} + E_k^-H_{ij}),$$
$$\hat{A}_k(\alpha) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j(\bar{A}_{ij} + \bar{B}_i(E_kF_{ij} + E_k^-H_{ij})).$$

3 Estimation of domain of attraction

Construct a fuzzy Lyapunov function:

$$V(\xi)=\xi^{\mathrm{T}}P(\alpha)\xi, \tag{12}$$

where

$$P(\alpha) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_jP_{ij}.$$

Theorem 1 Consider the system (11). If there exist the matrices $X, H_{ij}, P_{ij} > 0, i,j = 1,2,\cdots,r$, with appropriate dimensions, satisfying

$$\begin{bmatrix} P_{ij} & \hat{A}_{ijk}^{\mathrm{T}}X^{\mathrm{T}} \\ X\hat{A}_{ijk} & X^{\mathrm{T}}+X-P_{i'j'}\end{bmatrix} > 0, \tag{13}$$

$\forall i,j,i',j' \in [1,r], k \in [1,2^m]$, and $\Omega(P(\alpha),\rho) \subset \bigcap\limits_{i,j=1}^{r}\mathcal{L}(H_{ij})$, then the closed-loop system (11) is asymptotically stable at the origin of the system with $\Omega(P(\alpha),\rho)$ contained in the domain of attraction.

Proof If the inequality (13) is feasible, then

$$X^{\mathrm{T}}+X-P_{i'j'}>0$$

for all $i',j'\in[1,r]$. So, X is the nonsingular matrix.

Choose the Lyapunov candidate (12), then $\forall\xi \in L_V(\rho) \subset \bigcap\limits_{i,j=1}^{r}\mathcal{L}(H_{ij})$,

$$\Delta V(\xi) = V(\xi^+) - V(\xi) = \sum_{k=1}^{2^m}\eta_k\xi^{\mathrm{T}}(\hat{A}_k^{\mathrm{T}}(\alpha)P(\alpha^+)\hat{A}_k(\alpha) - P(\alpha))\xi = \sum_{k=1}^{2^m}\eta_k\xi^{\mathrm{T}}((\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\hat{A}_{ijk})^{\mathrm{T}} \times (\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i^+\alpha_j^+P_{ij})(\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\hat{A}_{ijk}) - (\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_jP_{ij}))\xi. \tag{14}$$

If

$$\hat{A}_k^{\mathrm{T}}(\alpha)P(\alpha^+)\hat{A}_k(\alpha) - P(\alpha) < 0, \forall k \in [1,2^m], \tag{15}$$

then $\Delta V(\xi) < 0$.

With (15), it follows that

$$\hat{A}_k^{\mathrm{T}}(\alpha)X^{\mathrm{T}}(X^{\mathrm{T}})^{-1}P(\alpha^+)X^{-1}X\hat{A}_k(\alpha)-P(\alpha)<0, \forall k \in [1,2^m]. \tag{16}$$

From Schur complement lemma and the inequality (16), we can get

$$\begin{bmatrix} P(\alpha) & \hat{A}_k^{\mathrm{T}}(\alpha)X^{\mathrm{T}} \\ X\hat{A}_k(\alpha) & XP^{-1}(\alpha^+)X^{\mathrm{T}}\end{bmatrix} >0, \forall k\in[1,2^m]. \tag{17}$$

With the inequality $(P(\alpha^+)-X)^{\mathrm{T}}P^{-1}(\alpha^+)(P(\alpha^+) - X) \geqslant 0$, we obtain

$$X^{\mathrm{T}}P^{-1}(\alpha^+)X \geqslant X^{\mathrm{T}} + X - P(\alpha^+). \tag{18}$$

Thus, with (17) and (18), it follows that

$$\begin{bmatrix} P(\alpha) & \hat{A}_k^{\mathrm{T}}(\alpha)X^{\mathrm{T}} \\ X\hat{A}_k(\alpha) & X^{\mathrm{T}}+X-P(\alpha^+)\end{bmatrix} >0, \forall k\in[1,2^m]. \tag{19}$$

Therefore,

$$\begin{bmatrix} \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r}\alpha_i\alpha_jP_{ij} & (*)^{\mathrm{T}} \\ X\big(\sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r}\alpha_i\alpha_j\hat{A}_{ijk}\big) & \varDelta\end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{i'=1}^{r}\sum_{j'=1}^{r}\alpha_i\alpha_j\alpha_{i'}^+\alpha_{j'}^+ \cdot \begin{bmatrix} P_{ij} & \hat{A}_{ijk}^{\mathrm{T}}X^{\mathrm{T}} \\ X\hat{A}_{ijk} & X^{\mathrm{T}}+X-P_{i'j'}\end{bmatrix}>0, \tag{20}$$

for any $k \in [1,2^m]$, where

$$\varDelta = X^{\mathrm{T}} + X - \sum_{i'=1}^{r}\sum_{j'=1}^{r}\alpha_{i'}^+\alpha_{j'}^+P_{i'j'}.$$

If the inequality (13) holds for any $i,j,i',j' \in [1,r], k \in [1,2^m]$, then $\Delta V(\xi) < 0$ for any $\xi \in L_V(\rho)\setminus\{0\}$. We can conclude that the system (11) is asymptotically stable at the origin, and $\Omega(P(\alpha),\rho)$ is contained in the domain of attraction. The proof is completed.

Theorem 1 gives a sufficient condition for the level set $L_V(\rho)$ to be inside the domain of attraction. In order to obtain the least conservative estimation of domain of attraction, we may choose the “largest” one of the level sets satisfying the conditions of Theorem1. This problem can be formulated as the following constrained optimization problem:

$$\max_{P(\alpha)>0,\rho,H_{ij}} \lambda, \tag{21}$$
$$\text{s.t. a) } \lambda\chi_{\mathrm{R}} \subset \Omega(P(\alpha),\rho)$$
$$\text{b) the inequality (13)},$$
$$\text{c) } \Omega(P(\alpha),\rho) \subset \bigcap_{i,j=1}^{r} \mathcal{L}(H_{ij}).$$

The shape reference set χ_{R} may be chosen to be a polyhedron, i.e.

$$\chi_{\mathrm{R}} = \mathrm{co}\{\xi_1, \xi_2, \cdots, \xi_p\}. \tag{22}$$

The set χ_{R} may also be chosen to be an ellipsoid, i.e.

$$\chi_{\mathrm{R}} = \{\xi \in \mathbb{R}^{n+n_c} : \xi^{\mathrm{T}} R\xi \leqslant 1, R > 0\}. \tag{23}$$

Since it is obvious that $\bigcap_{i,j=1}^{r} \Omega(P_{ij},\rho) \subset \Omega(P(\alpha),\rho)$, we choose $\bigcap_{i,j=1}^{r} \Omega(P_{ij},\rho)$ to be the estimation of the set $\Omega(P(\alpha),\rho)$. Then, the optimization problem (21) can be formulated as the following optimization problem:

$$\max_{P_{ij}>0,\rho,H_{ij}} \lambda, \tag{24}$$
$$\text{s.t. a) } \lambda\chi_{\mathrm{R}} \subset \Omega(P_{ij},\rho), \forall i,j \in [1,r]$$
$$\text{b) the inequality (13)},$$
$$\text{c) } |h_{ij(s)}\xi| \leqslant 1, \forall \xi \in \Omega(P_{ij},\rho),$$
$$\forall i,j \in [1,r], s \in [1,m].$$

Let

$$Q = \Big(\frac{X}{\rho}\Big)^{-\mathrm{T}}, Z_{ij} = H_{ij}Q,$$
$$z_{ij(s)} = h_{ij(s)}Q, Q_{ij} = Q^{\mathrm{T}}\Big(\frac{P_{ij}}{\rho}\Big)Q,$$
$$\forall i,j \in [1,r], s \in [1,m],$$

where $h_{ij(s)}, z_{ij(s)}$ is the s-th row of the matrices H_{ij} and Z_{ij}, respectively.

Since $(Q_{ij} - Q)^{\mathrm{T}} Q_{ij}^{-1}(Q_{ij} - Q) \geqslant 0$, we have

$$QQ_{ij}^{-1}Q^{\mathrm{T}} \geqslant Q^{\mathrm{T}} + Q - Q_{ij},$$
$$\forall i,j \in [1,r].$$

If the shape reference set χ_{R} is a polyhedron as defined in (22), then Constraint a) is equivalent to

$$\lambda^{-2}\xi_q^{\mathrm{T}}\Big(\frac{P_{ij}}{\rho}\Big)\xi_q \leqslant 1 \Leftrightarrow$$
$$\begin{bmatrix} \lambda^{-2} & \xi_q^{\mathrm{T}} \\ \xi_q & (\frac{P_{ij}}{\rho})^{-1} \end{bmatrix} \geqslant 0 \Leftrightarrow$$
$$\begin{bmatrix} \lambda^{-2} & \xi_q^{\mathrm{T}} \\ \xi_q & Q(Q^{\mathrm{T}}(\frac{P_{ij}}{\rho})Q)^{-1}Q^{\mathrm{T}} \end{bmatrix} \geqslant 0 \Leftrightarrow$$
$$\begin{bmatrix} \lambda^{-2} & \xi_q^{\mathrm{T}} \\ \xi_q & QQ_{ij}^{-1}Q^{\mathrm{T}} \end{bmatrix} \geqslant 0 \Leftarrow$$
$$\begin{bmatrix} \lambda^{-2} & \xi_q^{\mathrm{T}} \\ \xi_q & Q^{\mathrm{T}} + Q - Q_{ij} \end{bmatrix} \geqslant 0.$$

The inequality (13) is equivalent to

$$\begin{bmatrix} X^{-1}P_{ij}X^{-\mathrm{T}} & X^{-1}\hat{A}_{ijk} \\ \hat{A}_{ijk}X^{-\mathrm{T}} & X^{-\mathrm{T}} + X^{-1} - X^{-1}P_{i'j'}X^{-\mathrm{T}} \end{bmatrix} > 0 \Leftrightarrow$$
$$\begin{bmatrix} Q_{ij} & (*)^{\mathrm{T}} \\ \bar{A}_{ij}Q + \bar{B}_i(E_kF_{ij}Q + E_k^{-}Z_{ij}) & Q^{\mathrm{T}} + Q - Q_{i'j'} \end{bmatrix} > 0,$$
$$\forall i,j,i',j' \in [1,r], k \in [1,2^m].$$

Constraint c) is equivalent to

$$h_{ij(s)}\Big(\frac{P_{ij}}{\rho}\Big)^{-1}h_{ij(s)}^{\mathrm{T}} \leqslant 1 \Leftrightarrow$$
$$h_{ij(s)}Q\Big(Q^{\mathrm{T}}\Big(\frac{P_{ij}}{\rho}\Big)Q\Big)^{-1}Q^{\mathrm{T}}h_{ij(s)}^{\mathrm{T}} \leqslant 1 \Leftrightarrow$$
$$\begin{bmatrix} 1 & h_{ij(s)}Q \\ Q^{\mathrm{T}}h_{ij(s)}^{\mathrm{T}} & Q_{ij} \end{bmatrix} \geqslant 0 \Leftrightarrow$$
$$\begin{bmatrix} 1 & z_{ij(s)} \\ z_{ij(s)}^{\mathrm{T}} & Q_{ij} \end{bmatrix} \geqslant 0,$$
$$\forall i,j \in [1,r], k \in [1,2^m], s \in [1,m].$$

Then, the optimization problem (24) can be modified as the following optimization problem:

$$\min_{Q>0,Q_{ij},Z_{ij}} \gamma, \tag{25}$$
s.t.
$$\text{a) } \begin{bmatrix} \gamma & \xi_q^{\mathrm{T}} \\ \xi_q & Q^{\mathrm{T}} + Q - Q_{ij} \end{bmatrix} \geqslant 0,$$
$$\text{b) } \begin{bmatrix} Q_{ij} & (*)^{\mathrm{T}} \\ \bar{A}_{ij}Q + \bar{B}_i(E_kF_{ij}Q + E_k^{-}Z_{ij}) & Q^{\mathrm{T}} + Q - Q_{i'j'} \end{bmatrix} > 0,$$
$$\text{c) } \begin{bmatrix} 1 & z_{ij(s)} \\ z_{ij(s)}^{\mathrm{T}} & Q_{ij} \end{bmatrix} \geqslant 0,$$
$$\forall i,j,i',j' \in [1,r], k \in [1,2^m], q \in [1,p], s \in [1,m],$$

where $\gamma = \lambda^{-2}$.

4 Anti-windup compensation design

From the optimization problem (25), we can see that the constraint b) cannot be an LMI in $E_{\mathrm{c}}, Q, Z_{ij}(i,j = 1,2,\cdots,r)$ simultaneously. This implies that we cannot obtain the anti-windup compensation gain E_{c} by directly solving an LMI optimization problem. However, with all H_{ij} and X fixed, the anti-windup compensation gain $E_{\rm c}$ can be solved by an LMI optimization problem as follows

$$\min_{P_{ij},E_{\rm c}} \gamma, \tag{26}$$

s.t.

a) $(\xi_q^0)^{\rm T}P_{ij}\xi_q^0-\gamma\leqslant 0$,

b) $\begin{bmatrix} P_{ij} & (*)^{\rm T} \\ X\bar{A}_{ij}+X\bar{B}_i(E_kF_{ij}+E_k^-H_{ij}) & X^{\rm T}+X-P_{i'j'}\end{bmatrix}>0$,

c) $P_{ij}\geqslant\gamma(h_{ij(s)})^{\rm T}h_{ij(s)}$,

$\forall i,j,i',j'\in[1,r],k\in[1,2^m],q\in[1,p],s\in[1,m]$.

Let

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\rm T} & X_{22}\end{bmatrix}, X_1=\begin{bmatrix} X_{11} \\ X_{12}^{\rm T}\end{bmatrix}, X_2=\begin{bmatrix} X_{12} \\ X_{22}\end{bmatrix},$$
$$X_{11}\in\mathbb{R}^{n\times n}, X_{12}\in\mathbb{R}^{n\times n_{\rm c}}, X_{22}\in\mathbb{R}^{n_{\rm c}\times n_{\rm c}},$$

then the optimization problem (26) is equivalent to

$$\min_{P_{ij},E_{\rm c}} \gamma, \tag{27}$$

s.t. a) $(\xi_q^0)^{\rm T}P_{ij}\xi_q^0-\gamma\leqslant 0$,

b) $\begin{bmatrix} P_{ij} & \varLambda^{\rm T} \\ \varLambda & X^{\rm T}+X-P_{i'j'}\end{bmatrix}>0$,

c) $P_{ij}\geqslant\gamma(h_{ij(s)})^{\rm T}h_{ij(s)}$,

$\forall i,j,i',j'\in[1,r],k\in[1,2^m],q\in[1,p],s\in[1,m]$,

where $\varLambda=X\bar{A}_{ij}+(X_1B_i+X_2E_{\rm c})(E_kF_{ij}+E_k^-H_{ij})$.

Therefore, we give an iterative optimization algorithm for designing the anti-windup compensation gain $E_{\rm c}$ such that the domain of attraction of the closed-loop system is as large as possible.

Algorithm 1 Iterative algorithm for determining anti-windup compensation gain $E_{\rm c}$:

Step 1 Given the initial reference set $\chi_{\rm R}=\text{co}\{\xi_1^0,\xi_2^0,\cdots,\xi_p^0\}$ and $E_{\rm c}^0=0$, solve the optimization problem (25). Denote the solution as $(\gamma_0,Q^0,Q_{ij}^0,Z_{ij}^0)$, $i,j=1,2,\cdots,r$. Set $\chi_{\rm R}=\gamma_0^{-\frac{1}{2}}\chi_{\rm R}$.

Step 2 Set $E_{\rm c}$ with an initial value, $k=1$ and $\gamma_{\rm opt}=1$.

Step 3 Solve the optimization problem (25), then denote the solution as $(\gamma_k,Q,Q_{ij},Z_{ij})$, $i,j=1,2,\cdots,r$.

Step 4 Let $\gamma_{\rm opt}=\gamma_k\gamma_{\rm opt},\chi_{\rm R}=\gamma_k^{-\frac{1}{2}}\chi_{\rm R},X=(\dfrac{Q}{\rho})^{-\rm T},H_{ij}=Z_{ij}Q^{-1},P_{ij}=\rho Q^{-\rm T}Q_{ij}Q^{-1}$.

Step 5 IF $|\gamma_k-\gamma_{k-1}|<\delta$, where δ is a pre-determined tolerant bound, GOTO Step 7, ELSE GOTO Step 6.

Step 6 Solve the optimization problem (27), then set the solution as $E_{\rm c}$. Let $k=k+1$, GOTO Step 3.

Step 7 IF $\gamma_k\leqslant\gamma_0$, $E_{\rm c}$ is a feasible solution and STOP. ELSE, set $E_{\rm c}$ with another initial value and GOTO Step 2.

Remark 1 If the reference set is chosen to be a ellipsoid defined as (23), then the constraint condition $a)$ of the optimization problem (27) should be changed to

$$P_{ij}\leqslant\gamma R, i,j=1,2,\cdots,r.$$

Remark 2 The optimization result of Algorithm 1 depends on the initial value of the compensation gain $E_{\rm c}$. In general, we select several typical initial values $E_{\rm c}^0$, by using Algorithm 1, choose the smallest $\gamma_{\rm opt}$ and the corresponding $E_{\rm c}$ as the "optimal" solution of the optimization problem.

5 A numerical example

Consider the following fuzzy system subject to actuator saturation:

Rule 1 IF x_1 is about 0, THEN

$$\begin{cases} x(k+1)=A_1x(k)+B_1u(k), \\ y(k)=C_1x(k). \end{cases} \tag{28}$$

Rule 2 IF x_1 is about $\pm\dfrac{\pi}{2}(|x_1|<\dfrac{\pi}{2})$, THEN

$$\begin{cases} x(k+1)=A_2x(k)+B_2u(k), \\ y(k)=C_2x(k), \end{cases} \tag{29}$$

where

$$A_1=\begin{bmatrix} 1.5 & 0.5 \\ 0.3 & -1\end{bmatrix}, B_1=\begin{bmatrix} 10 \\ 1\end{bmatrix}, C_1=[5\ \ 1],$$
$$A_2=\begin{bmatrix} 1 & 0.4 \\ 0.4 & -1.2\end{bmatrix}, B_2=\begin{bmatrix} 10 \\ 0\end{bmatrix}, C_2=[5\ \ 1],$$

and $u_0=15$. The membership functions of Rule 1 and Rule 2 are

$$p_1=\cos x_1, p_2=1-p_1.$$

In the absence of input saturating, we design the following dynamic fuzzy controller:

Rule 1 IF x_1 is about 0, THEN

$$\begin{cases} \eta(k+1)=A_{{\rm c}(1)}\eta(k)+B_{{\rm c}(1)}y(k), \\ v(k)=C_{{\rm c}(1)}\eta(k), \end{cases} \tag{30}$$

Rule 2 IF x_1 is about $\pm\dfrac{\pi}{2}(|x_1|<\dfrac{\pi}{2})$, THEN

$$\begin{cases} \eta(k+1) = A_{\mathrm{c}(2)}\eta(k) + B_{\mathrm{c}(2)}y(k), \\ v(k) = C_{\mathrm{c}(2)}\eta(k), \end{cases} \tag{31}$$

where

$$A_{\mathrm{c}(1)} = A_{\mathrm{c}(2)} = \begin{bmatrix} -20 & 2.5 \\ 2 & -4 \end{bmatrix},$$

$$B_{\mathrm{c}(1)} = B_{\mathrm{c}(2)} = \begin{bmatrix} 2 \\ -0.06 \end{bmatrix},$$

$$C_{\mathrm{c}(1)} = C_{\mathrm{c}(2)} = [-1 \ \ 0.1].$$

We choose the initial reference set as $\chi_{\mathrm{R}} = [x^{\mathrm{T}}(0) \ \ \eta^{\mathrm{T}}(0)]^{\mathrm{T}}$, where

$$x(0) = [0.8 \ \ 0.5]^{\mathrm{T}}, \eta(0) = [0 \ \ 0]^{\mathrm{T}}.$$

In the absence of anti-windup compensation gain, i.e. $E_{\mathrm{c}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, applying the optimization problem (25), we can get $\lambda = 4.77 \times 10^4$. In the presence of anti-windup compensation gain, set the initial value to be $E_{\mathrm{c}}^0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$. Applying Algorithm 1, we can obtain

$$\lambda = 5.01 \times 10^4, \ E_{\mathrm{c}} = \begin{bmatrix} -39.52 \\ 23.23 \end{bmatrix}.$$

Obviously, the anti-windup compensation gain enlarges the domain of attraction of the closed-loop system.

We choose a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$, and set $P_{ij} = P$ and $Q_{ij} = Q_1(i,j = 1,2,\cdots,r)$ in the optimization problem (25). In the presence of anti-windup compensation gain, set the initial value to be $E_{\mathrm{c}}^0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$. By Algorithm 1, we get

$$\lambda = 4.46 \times 10^4, \ E_{\mathrm{c}} = \begin{bmatrix} -34.00 \\ 22.46 \end{bmatrix}.$$

It is easy to see that the quadratic Lyapunov function approach is more conservative than the fuzzy Lyapunov function method.

6 Conclusion

In this paper, we have presented the fuzzy Lyapunov function technique to enlarge the domain of attraction of the discrete-time T-S fuzzy system subject to actuator saturation. An iteration algorithm is provided to design the anti-windup compensation gain such that the domain of attraction is as large as possible. A numerical example illustrates that the proposed approach is effective.

参考文献(References):

[1] JOHANSSON M, RANTZER A. Computation of piecewise quadratic Lyapunov functions for hybrid systems[J]. *IEEE Transactions on Automatic Control*, 1998, 43(4): 555 – 559.

[2] CAO Y Y, LIN Z. Stability analysis of discrete-time systems with actuator saturation by a saturation-dependent Lyapunov function[J]. *Automatica*, 2003, 39(7): 1235 – 1241.

[3] WANG Y, CAO Y Y, SUN Y X. Stability analysis and anti-windup design for discrete-time systems by a saturation-dependent Lyapunov function approach[C] //*Proceedings of 16th International Federation of Automatic Control World Congress*. Prague: IFAC, 2005.

[4] HU T, GOEBEL R, TEEL A R, et al. Conjugate Lyapunov functions for saturated linear systems[J]. *Automatica*, 2005, 41(11): 1949 – 1956.

[5] HU T, LIN Z. Composite quadratic Lyapunov functions for constrained control systems[J]. *IEEE Transactions on Automatic Control*, 2003, 48(3): 440 – 450.

[6] HU T, TEEL A R, ZACCARIAN L. Stability and performance for saturated systems via quadratic and nonquadratic Lyapunov functions[J]. *IEEE Transactions on Automatic Control*, 2006, 51(11): 1770 – 1780.

[7] TANAKA K, HORI T, WANG H O. New parallel distributed compensation using time derivative of membership functions: a fuzzy Lyapunov approach[C] //*The 40th IEEE Conference on Decision and Control*. Orlando: IEEE, 2001: 3942 – 3947.

作者简介:

罗艳斌 (1978—), 女, 博士, 主要研究方向为约束控制、时滞系统和模糊控制, E-mail: ybluo@hdu.edu.cn, 本文通讯作者;

姜偕富 (1963—), 男, 研究员, 主要研究方向为时滞系统、鲁棒控制、模糊控制和网络控制, E-mail: jiangxf@hdu.edu.cn;

钟伟民 (1976—), 男, 博士, 主要研究方向为智能优化算法、机器学习和化工过程建模与优化, E-mail: wmzhong@ecust.edu.cn.