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Routh table test for stability of commensurate fractional degree polynomials and their commensurate fractional order systems

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Abstract

A Routh table test for stability of commensurate fractional degree polynomials and their commensurate fractional order systems is presented via an auxiliary integer degree polynomial. The presented Routh test is a classical Routh table test on the auxiliary integer degree polynomial derived from and for the commensurate fractional degree polynomial. The theoretical proof of this proposed approach is provided by utilizing Argument principle and Cauchy index. Illustrative examples show efficiency of the presented approach for stability test of commensurate fractional degree polynomials and commensurate fractional order systems. So far, only one Routh-type test approach [1] is available for the commensurate fractional degree polynomials in the literature. Thus, this classical Routh-type test approach and the one in [1] both can be applied to stability analysis and design for the fractional order systems, while the one presented in this paper is easy for peoples, who are familiar with the classical Routh table test, to use.

Keywords: Fractional order systems, stability, commensurate fractional degree polynomials, Routh table test

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1 Introduction

Fractional order systems (FOSs) have attracted increasing attention and gained growing development during the past few years due to the FOSs existence in physical experiments [2,3], and the development of flexible methodologies for easy study and favorable performance of fractional order control systems [1–5]. It is noticed that the research of fractional-order circuits and systems has been as an emerging interdisciplinary research area in the circuits and systems because of physical FOS behavior existing in real capacitors [3]. One core research subject of FOSs from the control viewpoint is the stability analysis and synthesis. In the research field of systems and control, synthesis methods have been fertilized by using fractional order system analysis to achieve flexible and robust controllers [3,4,6,7]. Particularly, commensurate FOSs (CFOSs) have become an important type of FOSs because CFOSs share similar structures to the integer order dynamic systems and maintain the characteristic of general FOSs [8].

Routh table test is well-known for the stability analysis and synthesis of control systems, which provides zeros distribution of the system characteristic function via finite steps of simple algebraic calculations. However, the classical Routh test is not directly applicable to FOSs. Thus, despite those substantial achievements of FOSs, the Routh table test [9,10], as the most useful stability test with the lowest calculation load for classical integer order systems, is rarely investigated for FOSs, except our recent work [1].

Moreover, the advanced root (zero) finding methods for any general polynomials with degrees greater than four are approximate methods as well-known from Galois theory as discussed in [1]. Even powerful Matlab tools may also give mistake roots (zeros) in the right-half plane for a stable integer polynomial [1]. However, the Routh table can give correct results for the root (zero) distribution of any general integer degree polynomials because it is a strictly mathematically proved method, not an approximation method. This fundamental merit is significant as Routh-type methods for polynomials.

Therefore, what is current status in the Routh-type tests for the commensurate fractional degree polynomials? Thus, let us briefly review [1] for it. The main contribution of reference [1] is as the first paper in the literature: 1) to ask if there are Routh-type tests and methods for general commensurate fractional degree polynomials (CFDPs); 2) to present the uniform Routh-type tests and formulas for zero distribution of CFDPs and integer degrees polynomials (IDPs); 3) to handle the singular cases easily and correctly as much better than the classical Routh table test; 4) to reveal the symmetric property of zero distribution in the second singular case for the CFDPs; and 5) the last but not least, to present the strict theoretical mathematical proof for their Routh-type table tests and methods. On the other hand, the Routh-type table tests in [1] need to check the sign change numbers of both head column and non-zero tail "column" (i.e., non-zero tail sequence) in the table.

Thus, it is natural to ask if there is a possible approach to check only the sign change number of the head column of the Routh-type table for CFOSs as the classical Routh table test. If it is possible, what kind of that Routh-type table will be?

Motivated by the above discussions, this paper derives a method such that the classical Routh test can be applicable to CFOSs. To be specific, given a CFDP, we propose an auxiliary integer degree polynomial (AIDP) and use the classical Routh test on this AIDP for the CFDP. Then the zero distribution of the original CFDP is obtained via quite simple calculations of the classical Routh test on the AIDP. Thus, we give an affirmative answer that the classical Routh table test can be extended for CFOSs by applying this new method on the AIDP. Also, this new method for CFOSs inherits the merits of the classical Routh test for integer order systems, revealing the relationship between the coefficients and zero distribution, and how these coefficients will impact the system stability. Note that such an analytical result is not shared by computing zeros numerically for the stability checking. On the other hand, it also inherits the same special needs of classical Routh test to treat special singular cases, i.e., the first type of singular cases and the second type of singular cases, even they happen together. In order to solve this well-known singular case problems in the classical Routh table, many other alternately modified methods are developed for it, e.g., in [1,11–13]. In these singular cases, the reference methods in [1] have their benefit as the simplest way to treat these special cases and with the fewest rows.

The main difference between the method proposed in this paper and the only comparable existing method in [1] is listed as follows. a) The proposed method is to use the classical Routh table test, while the method [1] uses the Routh-type table test for CFDPs; b) The proposed method uses the auxiliary integer degree polynomial (AIDP), while the method [1] does not use AIDP; c) The proposed method only checks the number of sign changes in the head column of the Routh table on the AIDP, while the method [1] needs to check the numbers of sign changes in the head column and the non-zero tail "column" (non-zero tail sequence) of its Routh-type table; d) The determination formula of the proposed method is easier than the one in [1]; e) On the other hand, the number of the rows in the table may be larger than the one in [1]; and f) Both methods are accurate with strict theoretical proofs for their respective objectives.

Thus, the main contribution and novelty of this paper are summarized as follows:

1) as the first paper in the literature to ask if there is a classical Routh test method still valid for general CFDPs;

2) to present an AIDP for the CFDPs and their stability problems;

3) as the first to advance the classical Routh table test on the proposed new AIDP for the CFDP and its stability problems;

4) to develop a new approach via the classical Routh table test for analyzing various systems stability and revealing the relationship between the polynomial data (commensurate order and coefficients) and their zero distributions for the system synthesis;

5) to have advantage of determining the stability of CFDPs via the Routh table as shown in [1], i.e., an easiest way to determine the stability, while the stability is the first important issue for all systems including the fractional order systems;

6) to have potential applications for fractional order system stability problems and in emerging new areas in view of the broad applications of classical Routh table test for various stability problems in science and engineering [14–16];

7) to present the theoretical proof for the classical Routh table test on the CFDP stability via a similar base proof approach as in [1]; and

8) to present an easy and accurate method for stability analysis of CFDPs and fractional order systems compared to all common existing methods which are via essential approximation way by taking direct roots/poles calculation or linear matrix inequality for stability analysis of general fractional degree polynomials and systems, except the method in [1].

Table 1 is for the notations of symbols used in this paper.

Table 1 Notations.

Symbol	Representation
$\mathbb{R}$	Real numbers
$\mathbb{R}_+$	Positive real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{I}$	Integer numbers
$\mathbb{N}$	Natural numbers
$\mathbb{N}_{\mathrm{odd}}$	Odd numbers
$\backslash$	Subtraction operator for sets
$\deg(f)$	Degree of polynomial $f(s)$
$\gcd(f,g)$	Greatest common divisor of f and g
$\Re(c)$	Real part of c ($c \in \mathbb{C}$)
$\Im(c)$	Imaginary part of c ($c \in \mathbb{C}$)
RPS	Riemann principal sheet
$n_{\mathrm{r}}(F)$	No. of function zeros of F in RHP of RPS
$n_{\mathrm{i}}(F)$	No. of function zeros of F on jω axis of RPS
$n_{\mathrm{l}}(F)$	No. of function zeros of F in LHP of RPS
$\langle\cdot\rangle$	Round-off operator, e.g., $\langle 4.5\rangle = 5$, $\langle 4.4\rangle = 4$

2 Preliminaries

In this section, we introduce preliminary background knowledge of this paper.

A fractional order integrator of degree α has the input-output relation as

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)\mathrm{d}\tau, \tag{1}$$

$$Y(s) = \frac{1}{s^\alpha}U(s) \tag{2}$$

as in time domain and frequency domain respectively, where $\alpha > 0$ and $\Gamma(\cdot)$ is the Gamma function. In contrast to the integer order one, the magnitude curve of a fractional order integrator has flexible slope as -20α dB/dec, determined by the fractional order α. In general, a linear time-invariant fractional order system with single input and single output has its transfer function [8] as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}}, \tag{3}$$

where $\alpha_n > \alpha_{n-1} > \ldots > \alpha_0 \geqslant 0$ and $\beta_m > \beta_{m-1} > \ldots > \beta_0 \geqslant 0$. Either the numerator or the denominator of $G(s)$ is a so-called fractional degree polynomial (FDP). System $G(s)$ is said to be of *commensurate order* if there exists $\alpha > 0$ such that $\alpha_k = k\alpha, \beta_l = l\alpha$ for $k = 1,\ldots,n$, and $l = 1,\ldots,m$. Correspondingly, its numerator and the denominator both are CFDPs. A general CFDP can be written as

$$F(s) = \sum_{k=0}^{n} c_k s^{k\alpha}, \tag{4}$$

where the commensurate $\alpha > 0$, and $k = 0, 1, \dots, n$. The definitions in (1)–(3) are used to define a fractional order system, where equation (4) may represent its system characteristic polynomial.

The Riemann surface RS is defined as $\mathrm{RS} \triangleq \{s| -\infty < \arg(s) < +\infty\} = \bigcup_{k\in\mathbb{I}} \mathrm{RS}_k$, where the kth Riemann sheet RS_k is defined as $\mathrm{RS}_k \triangleq \{s| -\pi + 2k\pi < \arg(s) \leqslant \pi + 2k\pi\}$ for any $k \in \mathbb{I}$. Particularly, the Riemann principal sheet RPS is defined as the central Riemann sheet as $\mathrm{RPS} \triangleq \mathrm{RS}_0 = \{s| -\pi < \arg(s) \leqslant \pi\}$, i.e., $k = 0$.

The transfer function $G(s)$ has its single-valued branch on each RS_k. Specifically, the branch in the RPS is utilized to describe the fractional order system, because it determines the Cauchy principal value of the integral corresponding to the inverse Laplace transformation of $G(s)$. The direct application of residue theorem implies that only the poles of $G(s)$ in the RPS determine the system stability and dynamic performance. In particular, the following lemma with respect to stability of $G(s)$ is well-known.

Lemma 1 [17, 18] The following two stability critetions hold.

Criterion 1: A fractional system (3) is BIBO stable if and only if its transfer function has no pole in the closed right half complex plane of the RPS, i.e., $n_{\mathrm{r}}(F) = 0$ and $n_{\mathrm{i}}(F) = 0$.

Criterion 2: In particular, let $D(s^\alpha) = F(s) = \sum_{k=0}^{n} c_k s^{k\alpha}$ be the denominator of the transfer function of a CFOS, where $D(\lambda) \triangleq \sum_{k=0}^{n} c_k \lambda^k$ (or $D(s) \triangleq \sum_{k=0}^{n} c_k s^k$) is an integer degree polynomial (IDP), and $\alpha > 0$ is the commensurate order. Then the system is stable if $\alpha < 2$ and $|\arg(\lambda_i)| > \frac{\pi\alpha}{2}$, where λ_i is any zero of $D(\lambda) = D(s^\alpha) = 0$.

Because this paper focuses on the CFOSs stability via their characteristic polynomials, i.e., CFDPs, therefore different from most existing works based on Criterion 2, only [1] and our test for zeros distribution are associated with Criterion 1 in Lemma 1. Especially, here our proposed method is via the classical Routh table test on the associated AIDP for the CFDP, making its uniqueness.

We describe the key preliminaries as summarized below as the Cauchy index and two lemmas, which will be used to prove our main result Theorem 1 in Section 3.

Definition 1 (Cauchy index [9]) Let $f(x)$ and $g(x)$ be two real continuous functions and have finite number of zeros in $\mathbb{R}$. As x changes from a to b (real numbers or $\pm\infty$), denote $\mathcal{J}_1$ and $\mathcal{J}_2$ as the jumps numbers of $\frac{g(x)}{f(x)}$ (or $\frac{f(x)}{g(x)}$) from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$, respectively. Then the Cauchy index of $\frac{g(x)}{f(x)}$ (or $\frac{f(x)}{g(x)}$) from a to b is defined as

$$I_a^b \frac{g(x)}{f(x)} \triangleq \mathcal{J}_1 - \mathcal{J}_2 \ \ (\text{or } I_a^b \frac{f(x)}{g(x)} \triangleq \mathcal{J}_1 - \mathcal{J}_2). \tag{5}$$

Lemma 2 Consider a function $\Phi(x) \triangleq f(x) + jg(x) : \mathbb{R} \to \mathbb{C}$, where $f(x)$ and $g(x)$ are continuous real functions. If $\Phi(x) \neq 0$ for $x \in (a, b)$, then its net phase change $\Delta_a^b \arg(\Phi(x))$ as x changes from a to b can be calculated by using the Cauchy index as

$$\Delta_a^b \arg(\Phi(x)) = -\pi I_a^b \frac{g(x)}{f(x)} + \lim_{x\to b^-} \arctan \frac{g(x)}{f(x)} - \lim_{x\to a^+} \arctan \frac{g(x)}{f(x)}, \tag{6}$$

or

$$\Delta_a^b \arg(\Phi(x)) = \pi I_a^b \frac{f(x)}{g(x)} + \lim_{x\to b^-} \operatorname{arccot} \frac{f(x)}{g(x)} - \lim_{x\to a^+} \operatorname{arccot} \frac{f(x)}{g(x)}. \tag{7}$$

Proof The phase function can be represented as $\arg(\phi(x)) = \arctan \frac{g(x)}{f(x)}$. If $\arg(\phi(x))$ is continuous in the region $x \in (a, b)$, then

$$\Delta_a^b \arg(\Phi(x)) = \lim_{x\to b^-} \arctan \frac{g(x)}{f(x)} - \lim_{x\to a^+} \arctan \frac{g(x)}{f(x)}. \tag{8}$$

Otherwise, $\arg(\Phi(x))$ is discontinuous at some points in the region $x \in (a, b)$. To be specific, as x passes a discontinuous point $c \in (a, b)$, the phase $\arg(\Phi(x))$ will have a change of π. Then we can use Cauchy index from a to b to identify its phase change as

$$\Delta_a^b \arg(\Phi(x)) - (\lim_{x\to b^-} \arctan \frac{g(x)}{f(x)} - \lim_{x\to a^+} \arctan \frac{g(x)}{f(x)}) = \pi(\mathcal{J}_2 - \mathcal{J}_1) = -\pi I_a^b \frac{g(x)}{f(x)}, \tag{9}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are the jump numbers as in Definition 1. Thus equation (6) holds. Note that the phase function can be also represented as $\arg(\Phi(x)) = \operatorname{arccot} \dfrac{f(x)}{g(x)}$, then equation (7) holds similarly to (6). □

Next we brief famous Routh table as a Lemma 3 here. It refers to [9,10].

Lemma 3 (Routh, [9,10]) Consider a real coefficient integer degree polynomial $P(s)$ as

$$P(s) = a_m s^m + b_m s^{m-1} + a_{m-1} s^{m-2} + b_{m-1} s^{m-3} + \dots, \quad (10)$$

where m is a positive integer number. Then $n_{\mathrm{r}}(P)$ and $n_{\mathrm{i}}(P)$ can be determined by the Routh table test on $P(s)$ as follows:

i) The number $n_{\mathrm{i}}(P)$ is equal to the number of real zeros of $d(x) = d(\omega) \triangleq \gcd(\Re(P(\mathrm{j}\omega)), \Im(P(\mathrm{j}\omega)))$.

ii) The following equality with respect to Cauchy index holds:

$$\begin{aligned} &\mathcal{I}_{-\infty}^{+\infty} \frac{b_m x^{m-1} - b_{m-1} x^{m-3} + b_{m-2} x^{m-5} - \dots}{a_m x^m - a_{m-1} x^{m-2} + a_{m-2} x^{m-4} - \dots} \\ &= m - 2n_{\mathrm{r}}(P) - n_{\mathrm{i}}(P). \end{aligned} \quad (11)$$

Remark 1 Lemma 3 basically states that the zeros distribution of integer degree polynomial $P(s)$ in (10) can be calculated via the Routh test, which is a well-known result. Lemma 3 presents an important analytical tool for the zeros distribution via the Cauchy index.

The key features of the proposed approach in this paper are as the same as the classical Routh table test has. Thus, it is easy for readers, who are familiar with the classical Routh test, to use it.

3 Routh test for CFDPs

In this section, we present the result that advances the classical Routh test for CFDPs as Theorem 1.

Theorem 1 Consider a commensurate fractional degree polynomial function $F(s)$ as

$$F(s) = \sum_{k=0}^{n} c_k s^{k\alpha}, \quad (12)$$

where its coefficients $c_0, c_1, \dots, c_n \in \mathbb{R}$ and $c_0 \neq 0, c_n \neq 0$, and the commensurate $\alpha > 0$. Construct an auxiliary integer degree polynomial $P(s)$ from $F(s)$ as

$$P(s) = \sum_{k=0}^{2n} p_k s^k, \quad (13)$$

where the coefficients of $P(s)$ are

$$\begin{cases} p_{2k} = c_k \cdot (-1)^{n-k+1} \cos(\pi k\alpha/2), \\ p_{2k-1} = c_k \cdot (-1)^{n-k} \sin(\pi k\alpha/2), \end{cases} \quad k = 0, 1, \dots, n. \quad (14)$$

Then we have

$$n_{\mathrm{i}}(F) = n_{\mathrm{i}}(P), \quad (15)$$

$$n_{\mathrm{r}}(F) = n_{\mathrm{r}}(P) + \langle n\alpha/2 \rangle - n, \quad (16)$$

where $n_{\mathrm{i}}(F)$ and $n_{\mathrm{r}}(F)$ are the numbers of zeros of the CFDP $F(s)$ on the imaginary axis and in the right-half-plane (RHP) of the RPS respectively, while $n_{\mathrm{i}}(P)$ and $n_{\mathrm{r}}(P)$ are numbers of zeros of the AIDP $P(s)$ on the imaginary axis and in the RHP of the complex plane respectively, and $\langle \cdot \rangle$ is a round-off operator as $x = m + \gamma, m \in \mathbb{N}, 0 \leqslant \gamma < 1$, and

$$\langle x \rangle = \begin{cases} m+1, & \text{if } \gamma \geqslant 0.5, \\ m, & \text{if } \gamma < 0.5. \end{cases} \quad (17)$$

Proof The approach of the proof is via the Argument Principle (to calculate zeros numbers in the RHP and imaginary axis and phase change of a complex function), the Cauchy index and Lemmas 2 and 3. The considered zeros numbers are $n_{\mathrm{i}}(\cdot)$ on the imaginary axis and $n_{\mathrm{r}}(\cdot)$ in the RHP of the RPS. The Argument Principle takes the Nyquist path as shown in Fig. 1 as Γ_1 and $-\Gamma_2$. Notice that the imaginary roots are the roots of the gcd of the real part and the imaginary part of the complex polynomial function. The goal is to prove $n_{\mathrm{i}}(F) = n_{\mathrm{i}}(P)$, and to present $n_{\mathrm{r}}(F)$ by $n_{\mathrm{r}}(P)$ and its fractional factor.

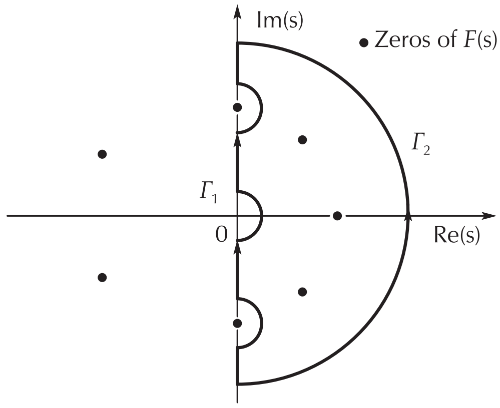

Fig. 1 Contour of the right half plane in RPS.

The proof consists of the following four steps. The first step is to build the real part and imaginary part of $F(s)$ as $s = \mathrm{j}\omega$. The second step is to show $n_{\mathrm{i}}(F) = n_{\mathrm{i}}(P)$ in (15). The third step is to show $n_{\mathrm{r}}(F) = n_{\mathrm{r}}(P) + \langle n\alpha/2 \rangle - n$ in

(16) when $n\alpha \notin \mathbb{N}_{\text{odd}}$. Then fourth step is to show (16) when the highest order $n\alpha \in \mathbb{N}_{\text{odd}}$.

Step 1 For $\omega \geqslant 0$, let

$$R(\omega) \triangleq \Re(F(\mathrm{j}\omega)) = \sum_{k=0}^{n} c_k \cos(k\pi\alpha/2)\omega^{k\alpha}, \tag{18}$$

$$I(\omega) \triangleq \Im(F(\mathrm{j}\omega)) = \sum_{k=0}^{n} c_k \sin(k\pi\alpha/2)\omega^{k\alpha}. \tag{19}$$

From the bijection transformation $x = \omega^\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, let

$$\bar{R}(x) \triangleq R(\omega) = \sum_{k=0}^{n} c_k \cos(k\pi\alpha/2)x^k, \tag{20}$$

$$\bar{I}(x) \triangleq I(\omega) = \sum_{k=0}^{n} c_k \sin(k\pi\alpha/2)x^k. \tag{21}$$

Moreover, let

$$\bar{d}(x) \triangleq \gcd(\bar{R}(x), \bar{I}(x)), \tag{22}$$

$$F_1(\mathrm{j}\omega) \triangleq \frac{F(\mathrm{j}\omega)}{\bar{d}(\omega^\alpha)} = \Big(\frac{\bar{R}(x)}{\bar{d}(x)} + \mathrm{j}\frac{\bar{I}(x)}{\bar{d}(x)}\Big)\Big|_{x=\omega^\alpha}. \tag{23}$$

Here the motivation to introduce the bijection transformation from ω to x in (20) and (21) is to absorb the fractional-order α, such that we have an integer-order polynomial in form of x instead of fractional order polynomials in (18) and (19) respectively. It is also helpful to compare it with the AIDP in (13).

Step 2 We take the Nyquist path as in Fig. 1 that encircles the entire right half plane and excludes pure imaginary zeros of $F(s)$ in (12). Applying the Argument Principle along this Nyquist path, we have

$$\begin{aligned} 2n_{\mathrm{r}}(F)\pi &= n\alpha\pi - \Delta_{\Gamma_1} \arg(F(s)) \\ &= n\alpha\pi - 2\Delta_{\Gamma_1^+} \arg(F(s)), \end{aligned} \tag{24}$$

where Γ_1 is the curve along the imaginary axis from $-j\infty$ to $+j\infty$ excluding the pure imaginary zeros of $F(s)$ as shown in Fig. 1, and Γ_1^+ is the half of Γ_1 from $j0$ to $+j\infty$. Note that $F(\mathrm{j}\omega)|_{\omega=0} = c_0 \neq 0$ and $F(s)$ is with real coefficients, i.e., having conjugate symmetric zeros, and the origin is not a zero of $F(s)$ and $F(\mathrm{j}\omega)$. Meanwhile, we have

$$\Delta_{\Gamma_1^+} \arg(F(s)) = \Delta_0^{+\infty} \arg(F_1(\mathrm{j}\omega)) + \frac{n_{\mathrm{i}}(F)}{2}\pi. \tag{25}$$

Substituting (25) into (24) yields

$$n_{\mathrm{r}}(F) = \frac{n\alpha}{2} - \frac{n_{\mathrm{i}}(F)}{2} - \frac{1}{\pi}\Delta_0^{+\infty} \arg(F_1(\mathrm{j}\omega)). \tag{26}$$

Furthermore, the imaginary zeros of $F(s)$ is $F(\mathrm{j}\omega) = 0$, if and only if $\bar{d}(x) = 0$ for $x = \omega^\alpha > 0$ as we notice the symmetric property and consider positive ω here. Thus, the number of the positive real zeros of $\bar{d}(x)$ is $n_{\mathrm{i}}(F)/2$.

From (20) and (21) of $F(s)$ and the coefficients (14) of $P(s)$, we have $n_{\mathrm{i}}(F) = n_{\mathrm{i}}(P)$ in (15) based on the statement (i) of Lemma 3, where $P(s)$ is with its $\alpha = 1$.

Step 3 Consider the case $n\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_{\text{odd}}$, i.e., $\deg(\bar{R}(x)) \geqslant \deg(\bar{I}(x))$. From Lemma 2 with equation (6), we have

$$\Delta_0^{+\infty} \arg(F_1(\mathrm{j}\omega)) = -\pi \mathcal{I}_0^{+\infty} \frac{\bar{I}(x)}{\bar{R}(x)} + \tau, \tag{27}$$

where

$$\begin{aligned} \tau &= \lim_{x\to+\infty} \arctan\frac{\bar{I}(x)}{\bar{R}(x)} - \lim_{x\to0^+} \arctan\frac{\bar{I}(x)}{\bar{R}(x)} \\ &= \lim_{x\to+\infty} \arctan\frac{\bar{I}(x)}{\bar{R}(x)}. \end{aligned} \tag{28}$$

Thus, $-\pi/2 < \tau < \pi/2$. Substituting (27) into (26) yields

$$n_{\mathrm{r}}(F) = \langle\frac{n\alpha}{2}\rangle - \frac{n_{\mathrm{i}}(F)}{2} + \mathcal{I}_0^{\infty}\frac{\bar{I}(x)}{\bar{R}(x)}. \tag{29}$$

On the other hand, from Lemma 3 with $P(s)$ in (10), we substitute its coefficients (14) into equation (11) that yields

$$\mathcal{I}_{-\infty}^{+\infty}\frac{-\bar{I}(x^2)}{x\bar{R}(x^2)} = 2n - 2n_{\mathrm{r}}(P) - n_{\mathrm{i}}(P). \tag{30}$$

Next, we show the following by the definition of Cauchy index:

$$\mathcal{I}_{-\infty}^{+\infty}\frac{\bar{I}(x^2)}{x\bar{R}(x^2)} = 2\mathcal{I}_0^{\infty}\frac{\bar{I}(x)}{\bar{R}(x)}. \tag{31}$$

First, since both $\bar{I}(x^2)/x\bar{R}(x^2)$ and $\bar{I}(x)/\bar{R}(x)$ are continuous at $x = 0$, thus,

$$\mathcal{I}_{0^-}^{0^+}\frac{\bar{I}(x^2)}{x\bar{R}(x^2)} = \mathcal{I}_{0^-}^{0^+}\frac{\bar{I}(x)}{\bar{R}(x)} = 0. \tag{32}$$

Second, the following is true from the Cauchy index

$$\begin{aligned} \mathcal{I}_{0^+}^{\infty}\frac{\bar{I}(x)}{\bar{R}(x)} &= \mathcal{I}_{0^-}^{-\infty}\frac{\bar{I}(x^2)}{\bar{R}(x^2)} = -\mathcal{I}_{-\infty}^{0^-}\frac{\bar{I}(x^2)}{\bar{R}(x^2)} \\ &= -\mathcal{I}_{-\infty}^{0^-}\frac{-\bar{I}(x^2)}{x\bar{R}(x^2)} = \mathcal{I}_{-\infty}^{0^-}\frac{\bar{I}(x^2)}{x\bar{R}(x^2)}. \end{aligned} \tag{33}$$

Third, we have

$$\mathcal{I}_{0+}^{\infty}\frac{\bar{I}(x)}{\bar{R}(x)}=\mathcal{I}_{0^+}^{\infty}\frac{\bar{I}(x^2)}{\bar{R}(x^2)}=\mathcal{I}_{0+}^{\infty}\frac{\bar{I}(x^2)}{x\bar{R}(x^2)}. \quad (34)$$

Then (32)–(34) lead to (31). Then substituting (30) and (31) into (29) yields (16)

$$n_{\mathrm{r}}(F)=\langle n\alpha/2\rangle-n+n_{\mathrm{r}}(P).$$

Step 4 Finally, we shall show (16) for the remaining case $n\alpha\in\mathbb{N}_{\mathrm{odd}}$, i.e., $\deg(\bar{R}(x))<\deg(\bar{I}(x))$. From (7) in Lemma 2, we have

$$\Delta_{0+}^{+\infty}\arg(F_1(\mathrm{j}\omega))=\pi\mathcal{I}_{0^+}^{\infty}\frac{\bar{R}(x)}{\bar{I}(x)}+\sigma, \quad (35)$$

where

$$\begin{aligned}\sigma&=\lim_{x\to+\infty}\operatorname{arccot}\frac{\bar{R}(x)}{\bar{I}(x)}-\lim_{x\to0^+}\operatorname{arccot}\frac{\bar{R}(x)}{\bar{I}(x)}\\&=\frac{\pi}{2}\operatorname{sgn}(\frac{\bar{R}(0^+)}{\bar{I}(0^+)}).\end{aligned} \quad (36)$$

Substituting (36) and (35) into (26) yields

$$n_{\mathrm{r}}(F)=\frac{n\alpha}{2}-\frac{n_{\mathrm{i}}(F)}{2}-\mathcal{I}_{0+}^{\infty}\frac{\bar{R}(x)}{\bar{I}(x)}-\frac{1}{2}\operatorname{sgn}(\frac{\bar{R}(0^+)}{\bar{I}(0^+)}). \quad (37)$$

Notice that $P(s)$ in (13) and (14) is of degree $2n-1$ with the highest degree term $c_n\sin(n\pi\alpha/2)s^{2n-1}$ when $n\alpha\in\mathbb{N}_{\mathrm{odd}}$. According to Lemma 3, we substitute coefficients in (14) into (11) and get

$$\mathcal{I}_{-\infty}^{+\infty}\frac{x\bar{R}(x^2)}{\bar{I}(x^2)}=2n-1-2n_{\mathrm{r}}(P)-n_{\mathrm{i}}(P). \quad (38)$$

As the above step 3, equation $n_{\mathrm{i}}(F)=n_{\mathrm{i}}(P)$ still holds as in (15). Now from the Cauchy index we have

$$\mathcal{I}_{-\infty}^{+\infty}\frac{x\bar{R}(x^2)}{\bar{I}(x^2)}=2\mathcal{I}_{0}^{\infty}\frac{\bar{R}(x)}{\bar{I}(x)}+\operatorname{sgn}(\frac{\bar{R}(0^+)}{\bar{I}(0^+)}). \quad (39)$$

Now, from (39), (38) and (37), we have (11), i.e.,

$$n_{\mathrm{r}}(F)=\langle n\alpha/2\rangle-n+n_{\mathrm{r}}(P)$$

for the case $n\alpha\in\mathbb{N}_{\mathrm{odd}}$. This completes the proof. □

Remark 2 According to Theorem 1, $n_{\mathrm{i}}(F)$ and $n_{\mathrm{r}}(F)$ of CFDP $F(s)$ in (12) can be obtained from $n_{\mathrm{i}}(P)$ and $n_{\mathrm{r}}(P)$ via the classical Routh table test on its AIDP $P(s)$ in (13) with (14).

Remark 3 Notice that the AIDP $P(s)$ from $F(s)$ is different from the IDP $D(s)$ from $F(s)$ in Lemma 1, where $D(s)$ is commonly considered in the literature.

Remark 4 Theorem 1 applies the classical Routh table test on the AIDPs for CFDPs. That is different from paper [1] where the developed Routh-type table test on the CFDPs directly. These two different approaches have their individual advantages and disadvantages respectively. The method presented here is to take the well-known classical Routh table test, that may be easy for peoples who are familiar with the classical Routh table test.

Remark 5 Because the classical Routh test gives analytical (or symbolic) expressions for $n_{\mathrm{i}}(P)$ and $n_{\mathrm{r}}(P)$ using the coefficients of $P(s)$, it also holds for $F(s)$ according to (15) and (16) in Theorem 1. Such a merit is usually preferred in the analysis and synthesis of a FOS, while those methods of computing the zeros numerically may not have this merit.

Remark 6 Routh test has close relation to other topics such as the zero distribution with respect to sector regions or the unit circle [19]. As a result, Theorem 1 may also serve as an intermediate tool for those problems with respect to fractional order systems.

4 Examples

Example 1 Consider the following CFDP function $F(s)$ with commensurate order $\alpha=1/3$ and $n=4$,

$$F(s)=s^{\frac{4}{3}}+5s+s^{\frac{2}{3}}+2s^{\frac{1}{3}}+1.$$

Its AIDP in (13) with (14) is

$$P(s)=\frac{1}{2}s^8+\frac{\sqrt{3}}{2}s^7-5s^5-\frac{1}{2}s^4+\frac{\sqrt{3}}{2}s^3+\sqrt{3}s^2-s-1.$$

The Routh table of the $P(s)$ is as follows as Table 2.

Table 2 Routh table for $P(s)$ in Example 1.

s^8	0.5	0	−0.5	$\sqrt{3}$	−1
s^7	$\sqrt{3}/2$	−5	$\sqrt{3}/2$	−1	
s^6	2.87	−1	2.31	−1	
s^5	−4.7	0.173	−0.7		
s^4	−0.89	1.88	−1		
s^3	−9.71	4.56			
s^2	1.46	−1			
s^1	−2.09				
s^0	−1				

Then $n_{\rm i}(P) = 0$ and $n_{\rm r}(P) = 3$ based on the above classical Routh table. By Theorem 1, we have

$$n_{\rm i}(F) = n_{\rm i}(P) = 0,$$
$$n_{\rm r}(F) = n_{\rm r}(P) + \langle n\alpha/2\rangle - n = 3 + 1 - 4 = 0.$$

Therefore, by Criterion 1 of Lemma 1, the CFDP F(s) is stable.

In fact, $F(s) = \prod_{k=1}^{4}(s^{\frac{1}{3}} - \lambda_k)$ where $\lambda_{1,2,3,4} = \{-4.8703, -0.4192, 0.1448 + {\rm j}0.6848, 0.1448 - {\rm j}0.6848\}$ and $\arg(\lambda_{1,2,3,4}) = \{\pi, \pi, 0.4337\pi, -0.4337\pi\}$. Therefore, it is stable in view of

$$|\arg(\lambda_k)| > \frac{\pi}{2}\alpha = \frac{\pi}{6}, \quad \forall\, k = 1, 2, 3, 4,$$

and Criterion 2 of Lemma 1. Then the set of zeros of $F(s)$ in the RPS is empty, because $\arg(s^{\frac{1}{3}}) \in (-\frac{1}{3}\pi, \frac{1}{3}\pi]$, as $s \in {\rm RPS} = (-\pi, \pi]$, is not a case here. It verifies that $n_{\rm i}(F) = 0$, and $n_{\rm r}(F) = 0$.

This example illustrates that Theorem 1 along with the classical Routh table test gives correct zeros distribution of $F(s)$ without calculating its exact zeros.

For comparison, we run the method in [1], which is only available Routh-type method in the literature. It leads to a Routh-type table as Table 3.

Table 3 Routh-type table for Example 1 [1].

$r1$	−0.5	0	0.5	$\sqrt{3}$	1
$r2$	$\sqrt{3}/2$	5	$\sqrt{3}/2$	1	
$r3$	−2.87	−1	−2.31	−1	
$r4$	−4.7	−0.173	−0.7		
$r5$	0.89	1.88	1		
$r6$	−9.71	−4.56			
$r7$	−1.46	−1			
$r8$	−2.09				
$r9$	1				

From [1], it is noticed that the head sign change number $V_{\rm f} = 5$, the tail sign change number $V_{\rm l} = 4$, $n_{\rm i}(F) = 0$ (no zero row in the table), and

$$n_{\rm r}(F) = \langle n\alpha/2\rangle - (V_{\rm f} - V_{\rm l}) = \langle 2/3\rangle - (5 - 4) = 0.$$

Thus, we have the same results from [1]. Since this example has no singular case, it shows that the proposed method is easier than [1].

Example 2 Consider a CFDP

$$F(s) = s^{\pi} + s^{\frac{2}{3}\pi} + s^{\frac{1}{3}\pi} + 1,$$

with an irrational commensurate order $\alpha = \pi/3$ and $n = 3$. The associated AIDP in high precision is

$$P(s) = -0.2206s^6 - 0.9754s^5 - 0.9890s^4 + 0.1477s^3 + 0.0741s^2 + 0.9973s + 1.$$

The Routh table of $P(s)$ is in Table 4.

Table 4 Routh table of $P(s)$ in Example 2.

s^6	−0.2206	−0.9890	0.0741	1
s^5	−0.9754	0.1477	0.9973	
s^4	−1.0224	−0.1515	1	
s^3	0.2922	0.0433		
s^2	$0\ (\epsilon)\ [-1.6622\times10^{-16}]$	1		
s^1	$0.0433\epsilon - 0.2922$			
s^0	1			

The accurate calculation shows that the first singular case happens at row of s^2, thus a small positive (or negative) number ϵ has to be used to replace the leading element 0. That leads to a table as above.

From the analysis on the above Routh table, we have $n_{\rm i}(P) = 0$ and $n_{\rm r}(P) = 3$. Thus, from Theorem 1, we have $n_{\rm i}(F) = n_{\rm i}(P) = 0$ and $n_{\rm r}(F) = n_{\rm r}(P) + \langle n\alpha/2\rangle - n = 3 + \langle\pi/2\rangle - 3 = 2$. Indeed,

$$F(s) = (s^{\frac{\pi}{3}} + 1)(s^{\frac{\pi}{3}} - {\rm e}^{{\rm j}\frac{\pi}{2}})(s^{\frac{\pi}{3}} - {\rm e}^{-{\rm j}\frac{\pi}{2}}).$$

Therefore, the set of zeros of $F(s)$ in the RPS is $\{{\rm e}^{{\rm j}1.5}, {\rm e}^{-{\rm j}1.5}\}$, which is in the RHP and verifies $n_{\rm r} = 2$. Therefore, $F(s)$ is unstable.

In addition, $F(s)$ can be written as $F(s) = H(s^{\frac{\pi}{3}})$, where $H(x) = x^3 + x^2 + x + 1$ and $H(x)$ or $H(s)$ is stable polynomial. However, $F(s) = H(s^{\frac{\pi}{3}})$ is unstable as Theorem 1 correctly states.

For comparison with [1], its Routh-type table is Table 5.

Table 5 Routh-type table for Example 2 [1].

$r1$	0.2206	−0.9890	−0.0741	1
$r2$	−0.9754	−0.1477	0.9973	
$r3$	1.0224	−0.1515	−1	
$r4$	0.2922	−0.0433		
$r5$	1			
$r6$	0.0433			

From [1], it is noticed that the head sign change number $V_{\rm f} = 2$, the tail sign change number $V_{\rm l} = 2$, no zero row with $d = 0$, $n_{\rm i}(F) = 0$, and

$$n_{\rm r}(F) = \langle n\alpha/2\rangle - (V_{\rm f} - V_{\rm l}) = \langle\pi/2\rangle - (2 - 2) = 2.$$

The same results are obtained.

However, it should be emphasized that row 5 has a leading 0, i.e., a first singular case happens. It shows that the method in [1] treats the singular cases as the easiest way and with the fewest rows in the literature.

5 Conclusions

This paper presents a new method to solve zeros distribution of a commensurate fractional degree polynomial by virtue of the classical Routh test on a proposed auxiliary integer degree polynomial.

This paper discusses the stability test for the CFDPs and their fractional order systems via Routh table, which can guarantee the system behavior stable in view point of both frequency domain and state space if it passes the Routh Table test. It is noticed that Routh table test can be used broadly in science and engineering, including system behaviors, e.g., [14–16]. It will be interesting to further study the system behavior in the state space along the Routh table approach as presented here.

Furthermore, the rigorous mathematical proof is presented via the Argument principle and Cauchy index. Its significance is to present a way to apply the classical Routh table test to the commensurate fractional degree polynomials and commensurate fractional order systems for their stability analysis and synthesis. The illustrative examples show the effectiveness of the presented method.

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